LECTURE NOTE

ON

MATHEMATICAL METHOD I

(MTS 201)

BY

ADEOSUN SAKIRU ABIODUN

E-mail: adeosunsakiru@gmail.com

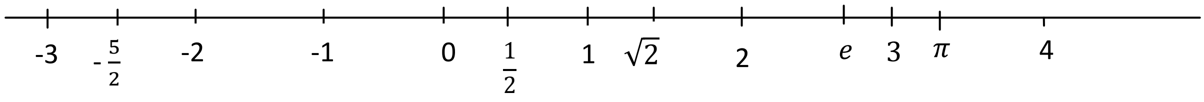
COURSE CONTENTS

Real-valued functions of a real variable. Review of differentiation, integration and application. Mean value theorem. Taylor series. Real valued functions of two or three variables (Functions of several variables). Jacobian functions, dependence and independence, Lagrange multiplier, multiple integrals, line integral,.

§ 1.0 REAL VALUED FUNCTIONS OF A REAL VARIABLE

INTRODUCTION

The collection of all real numbers is denoted by $\mathbb{R}$. Thus, $\mathbb{R}$ includes the integers$\cdots -2, -1, 0, 1, 2, \cdots$, the rational numbers, $\frac{p}{q}$, where p and q are integers $(q \neq 0)$, and the irrational numbers, like $\sqrt{2}$, π, e, etc. Members of $\mathbb{R}$ may be visualized as points on the real number line as shown in the figure below:



We write $a \in \mathbb{R}$ to mean a is a member of the set $\mathbb{R}$. In other words, a is a real number. Given two real numbers a and b with $a < b$, the closed interval $[a, b]$ consists of all x such that $a \leq x \leq b$ and the open interval (a, b) consists of all x such that $a < x < b$. Similarly, we may form the half open or clopen intervals $[a, b)$ and $(a, b]$.

The absolute value of a number $x \in \mathbb{R}$ is written as $|x|$ and is defined as

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

For example, $|2| = 2$, $|-2| = -(-2) = 2$. Some properties of $|x|$ are summarized as follows:

1. $|-x| = |x| \ \forall\, x \in \mathbb{R}$
2. $-|x| \leq x \leq |x| \ \forall\, x \in \mathbb{R}$
3. For a fixed $r > 0$, $|x| < r$ if and only if (iff) $x \in (-r, r)$
4. $\sqrt{x^2} = |x|,\ x \in \mathbb{R}$
5. (Triangle inequality) $|x + y| \leq |x| + |y| \ \forall\ x, y \in \mathbb{R}$

<u>Theorem 1.0.1</u>: If $a \geq 0$, then $|x| \leq a$ iff $-a \leq x \leq a$.

<u>Proof</u>: There are two statements to prove: first, that the inequality $|x| \leq a$ implies the inequalities $-a \leq x \leq a$ and conversely, that $-a \leq x \leq a$ implies $|x| \leq a$.

Suppose $|x| \leq a$. Then we also have $-a \leq -|x|$. But either $x = |x|$ or $x = -|x|$ and hence $-a \leq -|x| \leq x \leq |x| \leq a$. This proves the first statement.

Conversely, assume $-a \leq x \leq a$. Then if $x \geq 0$, we have $|x| = x \leq a$, whereas if $x \leq 0$, we have $|x| = -x \leq a$. In either case, we have $|x| \leq a$, and this complete the proof.

Theorem 1.0.2: (Triangle inequality) For any arbitrary real number x and y, we have $|x + y| \leq |x| + |y|$.

Proof: Adding the inequalities $-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$ we obtain

$$-(|x| + |y|) \leq x + y \leq |x| + |y|,$$

and hence, by Theorem 1.0.1, we conclude that

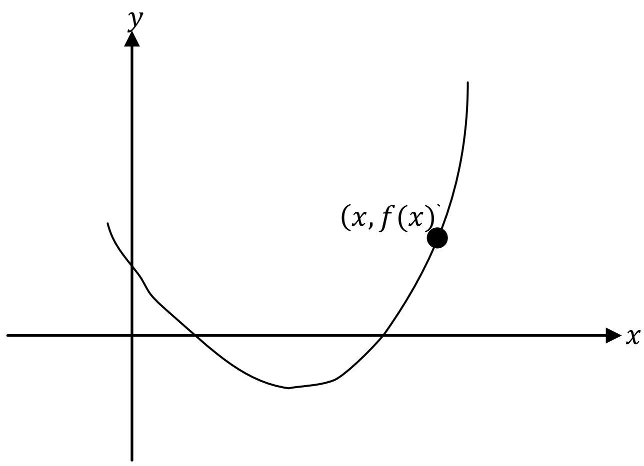
$$|x + y| \leq |x| + |y|. \qquad \blacksquare$$

If we take $x = a - c$ and $y = c - b$, then $x + y = a - b$ and the triangle inequality becomes $|a - b| \leq |a - c| + |b - c|$. This form of the triangle inequality is often used in practice.

1.1 FUNCTIONS

A **function** $f: A \to B$ is a rule that assigns to each $a \in$A one specific member $f(a)$ of B. The fact that the function f sends a to $f(a)$ is denoted symbolically by $a \mapsto f(a)$. For example, $f(x) = \frac{x^2}{(1-x)}$ assigns the number $\frac{x^2}{(1-x)}$ to each $x \neq 1$ in $\mathbb{R}$. We can specify a function f by giving the rule for $f(x)$. The set A is called the **domain** of f and B is the **codomain** of f. The **range** of f is the subset of B consisting of all the values of f. That is, the range of $f = \{f(x) \in B: x \in A\}$.

Given $f: A \to \mathbb{R}$. It means that f assigns a value $f(x)$ in $\mathbb{R}$ to each $x \in A$. Such a function is called a **real-valued function**. For a real-valued function $f: A \to \mathbb{R}$ defined on a subset A of $\mathbb{R}$, the graph of f consists of all the points $(x, f(x))$ in the xy –plane.



Let f be a function whose domain D_f and range R_f are sets of real numbers. Then f is said to be **even** if $f(x) = f(-x) \forall x \in D_f$. And f is said to **odd** if

$f(-x) = -f(x)\ \forall\, x \in D_f$. (Check whether $y = x^3 - 3x, x^4 - x^2$, $\sin x, \cos x$ are even or odd.) Also, f is said to be one-to-one if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2\ \forall\ x_1, x_2 \in D_f$.

A function f of the independent variables $x_1, x_2, \cdots, x_n$ with dependent variable y is of the form $y = f(x_1, x_2, \cdots, x_n)$. The function f is called **function of several variables**. If $n = 2$, we frequently write $z = f(x, y)$, and if $n = 3$, we write $w = f(x, y, z)$. In this case, the domain D of a function is the set of allowable input variables $(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, while the range is the set that contains all positive values for the output variable y. This means y is in the range of f if there exists ($\exists$) a $(x_1, x_2, \cdots, x_n) \in D$ so that $y = f(x_1, x_2, \cdots, x_n)$.

Remark: Some examples of real functions

1. Constant function. A function whose range consists of a single number is called a constant function. E.g. $f(x) = 5$.
2. Linear functions. A function g defined for all real x by a formula of the form $g(x) = ax + b$ is called a linear function because its graph is a straight line.
3. The Power function. For a fixed positive integer n, let f be defined by the equation $f(x) = x^n\ \forall$ real x. When $n = 1$, this is the identity function. When $n = 2$, the graph is parabola. For $n = 3$, the graph is a cubic curve.
4. Polynomial function. A polynomial function P is one defined for all real x by an equation of the form $P(x) = c_0 + c_2 x + c_3 x^2 + \cdots + c_n x^n = \sum_{k=0}^{n} c_k x^k$. The number $c_0, c_1, \ldots, c_n$ are called the coefficients of the polynomial and the nonnegative integer n is called its degree $(\text{if } c_n \neq 0)$.
5. Unit step function, $u(x)$, is defined as follow: $u(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x \geq 0 \end{cases}$
6. Signum function, sign (x), is defined as sign $(x) = \frac{x}{|x|} = \begin{cases} 1 & \text{if} \quad x > 0 \\ -1 & \text{if} \quad x < 0 \end{cases}$

1.2 LIMIT OF FUNCTIONS

We begin with a review of the concept of limits for real-valued functions of one variable. Recall that the definition of the limit of such functions is as follows.

Definition 1.2.1: Let $f: D \subset \mathbb{R} \to \mathbb{R}$ and let $c \in \mathbb{R}$. Then $\lim_{x \to c} f(x) = L$ means that for each $\epsilon > 0$ there exist some $\delta > 0$ such that that $|f(x) - L| < \epsilon$, whenever $0 < |x - c| < \delta$. (or $c < x < c + \delta$)

The two fundamental specific limits results which follow easily from the definitions are:

1. If $a \in \mathbb{R}$, then $\lim_{x\to c} a = a$ and
2. $\lim_{x\to c} x = c$ for any $c \in \mathbb{R}$.

The basic facts used to compute limits are contained in the following theorem.

Theorem 1.2.2: Suppose that for some real numbers L and M, $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$. Then

(i) $\lim_{x\to c} k = k$, where k is constant

(ii) $\lim_{x\to c}(f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$

(iii) $\lim_{x\to c}(f(x) - g(x)) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$

(iv) $\lim_{x\to c}(f(x)g(x)) = (\lim_{x\to c} f(x))(\lim_{x\to c} g(x))$

(v) $\lim_{x\to c}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ if $\lim_{x\to c} g(x) \neq 0$.

Proof:

(i) Let $f(x) = k \quad \forall\, x$ and $\epsilon > 0$ be given. Then $|f(x) - k| = |k - k| = 0 < \epsilon \quad \forall\, x.$ ■

For *(ii)* – *(iii)*, let $\epsilon > 0$ be given and let $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$. By definition $\exists\ \ \delta_1 > 0$ and $\delta_2 > 0$ such that

$|f(x) - L| < \frac{\epsilon}{3}$ whenever $0 < |x - c| < \delta_1$ (1)

$|g(x) - M| < \frac{\epsilon}{3}$ whenever $0 < |x - c| < \delta_2$ (2)

(ii) Let $\delta = \min(\delta_1, \delta_2)$. Then $0 < |x - c| < \delta$ implies that

$0 < |x - c| < \delta_1$ and $|f(x) - L| < \frac{\epsilon}{3}$ (by (1)) (3)

$0 < |x - c| < \delta_2$ and $|g(x) - M| < \frac{\epsilon}{3}$ (by (2)) (4)

Hence, if $0 < |x - c| < \delta$, then

$$\begin{aligned}\big|\big(f(x) + g(x)\big) - (L + M)\big| &= |(f(x) - L) + (g(x) - M)| \\ &\le |f(x) - L| + |g(x) - M| \quad \text{(Triangle inequality)} \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} \quad \text{(by (3) \& (4))} \\ &< \epsilon. \qquad ■\end{aligned}$$

(iii) Let δ be defined as in part *(ii).* Then $0 < |x - c| < \delta$ implies that

$$\begin{aligned}\big|\big(f(x) - g(x)\big) - (L - M)\big| &= |(f(x) - L) + (g(x) - M)| \\ &\le |f(x) - L| + |g(x) - M|\end{aligned}$$

$< \frac{\epsilon}{3} + \frac{\epsilon}{3}$

$< \epsilon \quad \blacksquare$

(iv) Let $\epsilon > 0$ be given. Let $\epsilon_1 = \min\left(1, \frac{\epsilon}{1+|L|+|M|}\right)$

Then $\epsilon_1 > 0$ and, by definition, there exists δ_1 and δ_2 such that

$|f(x) - L| < \epsilon_1$ whenever $0 < |x - c| < \delta_1$ (5)

$|g(x) - M| < \epsilon_2$ whenever $0 < |x - c| < \delta_2$ (6)

Let $\delta = \min(\delta_1, \delta_2)$. Then $0 < |x - c| < \delta$ implies that

$0 < |x - c| < \delta_1$ and $|f(x) - L| < \epsilon_1$ (by (5)) (7)

$0 < |x - c| < \delta_2$ and $|g(x) - M| < \epsilon_2$ (by (6)) (8)

Also,

$$
\begin{aligned}
|f(x)g(x) - LM| &= |(f(x) - L + L)(g(x) - M + M) - LM| \\
&= |(f(x) - L)(g(x) - M) + (f(x) - L)M + L(g(x) - M)| \\
&\le |f(x) - L||g(x) - M| + |f(x) - L||M| + |L||g(x) - M| \\
&< {\epsilon_1}^2 + |M|\epsilon_1 + |L|\epsilon_1 \\
&\le \epsilon_1 + |M|\epsilon_1 + |L|\epsilon_1 \\
&= (1 + |M| + |L|)\epsilon_1 \\
&< \epsilon \ . \ \blacksquare
\end{aligned}
$$

(v) Suppose that $M > 0$ and $\lim_{x\to c} g(x) = M$. Then we show that $\lim_{x\to c} \frac{1}{g(x)} = \frac{1}{M}$. Since $\frac{M}{2} > 0$, $\exists$ some $\delta_1 > 0$ such that

$|g(x) - M| < \frac{M}{2}$ whenever $0 < |x - c| < \delta_1$,

$-\frac{M}{2} + M < g(x) < \frac{3M}{2}$ whenever $0 < |x - c| < \delta_1$,

$0 < \frac{M}{2} < g(x) < \frac{3M}{2}$ whenever $0 < |x - c| < \delta_1$,

$\frac{1}{|g(x)|} < \frac{2}{M}$ whenever $0 < |x - c| < \delta_1$.

Let $\epsilon > 0$ be given. Let $\epsilon_1 = \frac{M^2\epsilon}{2}$. Then $\epsilon_1 > 0$ and there exists some $\delta > 0$ such that $\delta < \delta_1$ and $|g(x) - M| < \epsilon_1$ whenever $0 < |x - c| < \delta < \delta_1$,

$$
\begin{aligned}
\left|\frac{1}{g(x)} - \frac{1}{M}\right| &= \left|\frac{M - g(x)}{g(x)M}\right| \\
&= \frac{|g(x) - M|}{|g(x)M|} \\
&= \frac{1}{M} \cdot \frac{2}{M} \cdot \epsilon_1
\end{aligned}
$$

$= \frac{2\epsilon_1}{M^2}$

$= \epsilon$ whenever $0 < |x - c| < \delta$.

This complete the proof of the statement $\lim_{x \to c} \frac{1}{g(x)} = \frac{1}{M}$ whenever $M > 0$.

(iv) to prove *(v)* as follows:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \left(f(x) \cdot \frac{1}{g(x)} \right)$$

$= \lim_{x \to c} f(x) \cdot \lim_{x \to c} \left(\frac{1}{g(x)} \right)$

$= L \cdot \frac{1}{M} = \frac{L}{M}$. This complete the proof of the Theorem.

Now we take up the subjects of limits for real-valued functions of several variables.

Definition 1.2.3: Let $f: D \subset \mathbb{R}^n \to \mathbb{R}$, let $P_0 \in \mathbb{R}^n$ and let $L \in \mathbb{R}$. Then $\lim_{\substack{P \to P_0 \\ P \in D}} f(P) = L$ means that the distance, for each $\epsilon > 0$ there exists $\delta > 0$ such that if $P \in D$ and if $0 < |P - P_0| < \delta$, then $|f(P) - L| < \epsilon$.

Note that the first use vertical lines denotes absolute value while the second denotes distance between two points in $\mathbb{R}^n$. To begin computing limits we first need some specific results similar to those for functions of one variable. The basic principle is that if a function of more than one variable is considered as a function of more than one variable, then the limit of the function is computed by taking the limit of the function with respect to its only variable. One specific case of this principle is stated below.

Theorem1.2.4: Let $h: E \subset \mathbb{R} \to \mathbb{R}$ and set $f(x, y) = h(x)$. Suppose $\lim_{x \to a} h(x) = L$. Then $\lim_{(x,y) \to (a,b)} f(x, y) = L$ for any $b \in \mathbb{R}$.

For example, $\lim_{(x,y) \to (2,9)} x^3 = \lim_{x \to 2} x^3 = 8$ and $\lim_{(x,y) \to (2,9)} \sqrt{y} = \lim_{y \to 9} \sqrt{y} = 3$.

Essentially all examples of functions of several variables we will encounter are constructed from functions of one variable by addition, multiplication, division and composition. So the following Basic Limit Theorem will permit us to compute limits.

Theorem 1.2.5: Let $f, g: D \subset \mathbb{R}^n \to \mathbb{R}$. Suppose $\lim_{\substack{P \to P_0 \\ P \in D}} f(P) = L$ and $\lim_{\substack{P \to P_0 \\ P \in D}} g(P) = M$. Then

1. $\lim_{\substack{P \to P_0 \\ P \in D}} f(P) + g(P) = L + M$

2. $\lim_{\substack{P\to P_0 \\ P\in D}} f(P)g(P) = LM$ and
3. $\lim_{\substack{P\to P_0 \\ P\in D}} \frac{f(P)}{g(P)} = \frac{L}{M}$ provided $M \neq 0$.

Moreover, if $\lim_{\substack{P\to P_0 \\ P\in D}} f(P) = L = \lim_{\substack{P\to P_0 \\ P\in D}} g(P)$ and if $f(P) \leq h(P) \leq g(P)$, then $\lim_{\substack{P\to P_0 \\ P\in D}} h(P) = L$.

(The Sandwich Theorem for functions of several variables.)

1.2.6 Examples

(1) Suppose that $f(x) \leq g(x) \leq h(x)$ for all x in an open interval containing c and $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$. Then show that $\lim_{x\to c} g(x) = L$.

Proof: Let $\epsilon > 0$ be given. Then there exist $\delta_1 > 0,\ \delta_2 > 0$, and $\delta = \min\{\delta_1, \delta_2\}$ such that

$|f(x) - L| < \frac{\epsilon}{2}$ whenever $0 < |x - c| < \delta_1$

$|h(x) - L| < \frac{\epsilon}{2}$ whenever $0 < |x - c| < \delta_2$.

If $0 < |x - c| < \delta$, then $0 < |x - c| < \delta_1$, $0 < |x - c| < \delta_2$ and, hence,

$|g(x) - L| < \frac{\epsilon}{2}$ whenever $0 < |x - c| < \delta$, and $\lim_{x\to c} g(x) = L$.

(2) Evaluate each of the following limits.

(a) $\lim_{x\to 0}(x^2 - 6x + 3)$ (b) $\lim_{x\to 0} \frac{\sqrt{9+x}-3}{x}$ (c) $\lim_{x\to 2} \frac{x^2-x-2}{x^2-4}$ (d) $\lim_{x\to\infty} \frac{4x^3}{x^3+3}$.

Solution

(a) $\lim_{x\to 0}(x^2 - 6x + 3) = \lim_{x\to 0} x^2 - \lim_{x\to 0} 6x + \lim_{x\to 0} 3$

$= (0)^2 - 6(0) + 3$

$= 3$

(b) $\lim_{x\to 0} \frac{\sqrt{9+x}-3}{x} = \lim_{x\to 0} \left(\frac{\sqrt{9+x}-3}{x} \times \frac{\sqrt{9+x}+3}{\sqrt{9+x}+3}\right)$

$= \lim_{x\to 0} \left[\frac{x}{x(\sqrt{9+x}+3)}\right]$

$= \lim_{x\to 0} \frac{1}{\sqrt{9+x}+3}$

$= \frac{1}{\sqrt{9}+3}$

$= \frac{1}{6}$

(c) $\lim_{x\to 2} \frac{x^2-x-2}{x^2-4} = \lim_{x\to 2} \frac{(x+1)(x-2)}{(x+2)(x-2)}$

$= \lim_{x\to 2} \frac{x+1}{x+2}$

$$= \frac{\lim_{x\to 2}(x+1)}{\lim_{x\to 2}(x+2)}$$

$$= \frac{3}{4}$$

(d) $\lim_{x\to\infty} \frac{4x^3}{x^3+3} = \lim_{x\to\infty} \frac{\left(\frac{4x^3}{x^3}\right)}{\frac{x^3}{x^3}+\frac{3}{x^3}}$

$$= \lim_{x\to\infty} \frac{4}{1+\frac{3}{x^3}}$$

$$= \frac{\lim_{x\to\infty} 4}{\lim_{x\to\infty} 1+\lim_{x\to\infty}\frac{3}{x^3}}$$

$$= 4$$

Remark: There are some examples of limits which we can use L'Hospital rule and Taylor series to solve and we shall discuss these later.

1.3 CONTINUITY OF FUNCTIONS

Definition 1.3.1: (Continuity at a point) The function f is said to be continuous at c from the right if $f(c)$ is defined or exist, and $\lim_{x\to c^+} f(x) = f(c)$.

Definition 1.3.2: The function f is said to be continuous at c from the left if $f(c)$ is defined and $\lim_{x\to c^-} f(x) = f(c)$.

Definition 1.3.3: The function f is said to be (two sided) continuous at c if $f(c)$ is defined, and $\lim_{x\to c} f(x) = f(c)$.

Remark:

(1) The continuity definition requires that the following conditions be met if f is to be continuous at c (a point): (a) $f(c)$ is defined as a finite real number, (b) $\lim_{x\to c^-} f(x)$ exists and equals $f(c)$, (c) $\lim_{x\to c^-} f(x) = \mathrm{f(c)} = \lim_{x\to c^+} f(x)$. When a function f is not continuous at c, one or more, of these conditions are not met.

(2) All polynomials, $\sin x$, $\cos x$, e^x, $\sinh x$, $\cosh x$, b^x, $b \neq 1$ are continuous for all real values of x. All logarithmic functions, $\log_b x$, $b > 0, b \neq 1$ are continuous for all $x > 0$. Each rational function, $p(x)/q(x)$, is continuous where $q(x) \neq 0$.

(3) Alternative definition: Let $f: D \subset \mathbb{R} \to \mathbb{R}$ and let $a \in D$. Then f is continuous at a means $\lim_{\substack{x\to a \\ x\in D}} f(x) = f(a)$. For function of **several variables**: Let $f: D \subset \mathbb{R}^n \to \mathbb{R}$ and let $x_0 \in D$. Then f is continuous at x_0 means $\lim_{\substack{x\to x_0 \\ x\in D}} f(x) = f(x_0)$.

(4) **Epsilon definition:** We can define f as continuous at $x = x_0$ if for any $\epsilon > 0$ we can find $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$.

(5) Continuity in an interval: A function f is said to be continuous in an interval if it is continuous at ll points of the interval. In particular, if f is defined in the close interval $[a, b]$, then f is continuous in the interval if and only if $\lim_{x\to x_0} f(x) = f(x_0)$ for $a < x_0 < b$, $\lim_{x\to a^+} f(x) = f(a)$ and $\lim_{x\to b^-} f(x) = f(b)$.

1.3.4 Continuity Examples

(1) Verify the continuity of the following functions:

(a) $f(x) = \frac{x^2-1}{3x^2+2x-1}$ at $x = 1$

(b) $f(x) = |x|$ at $x = 0$.

<u>Solution:</u>

(a) $f(1) = \frac{1^2-1}{3(1^2)+2(1)-1} = 0$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2-1}{3x^2+2x-1}$$

$$= \lim_{x\to 1} \frac{(x-1)(x+1)}{(3x-1)(x+1)}$$

$$= \lim_{x\to 1} \frac{(x-1)}{(3x-1)}$$

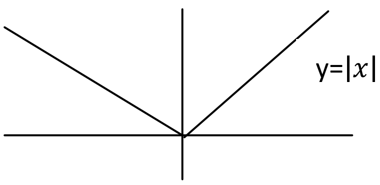
$$= \frac{0}{2}$$

$$= 0$$

$\therefore \lim_{x\to 1} f(x) = f(1) = 0.$

Hence, $f(x)$ is continuous at $x = 1$.

(b)



We have $f(0) = |0| = 0$

$\lim_{x\to 0^-} f(x) = 0$

$\lim_{x\to 0^+} f(x) = 0$

$\therefore f(x)$ is continuous at $x = 0$ since $\lim_{x\to 0} f(x) = f(0) = 0$.

[Alternatively, Let $\epsilon > 0$ be given. Let $\delta = \epsilon$. Then $|x - 0| < \epsilon \Rightarrow |x| < \epsilon$. Hence, $\lim_{x\to 0}|x| = 0$.]

(2) Show that the constant function $f(x) = 4$ is continuous at every real number c. Show that for every constant $k, f(x) = k$ is continuous at every real number c.

Solution:
First of all, if $f(x) = 4$, then $f(c) = 4$. We need to show that $\lim_{x\to c} 4 = 4$.

For each $\epsilon > 0$, let $\delta = 1$. Then $|f(x) - f(c)| = |4 - 4| = 0 < \epsilon$ for all x such that $|x - c| < 1$. Secondly, for each $\epsilon > 0$, let $\delta = 1$. Then $|f(x) - f(c)| = |k - k| = 0 < \epsilon$ for all x such that $|x - c| < 1$. ■

(3) Show that $f(x) = 3x - 4$ is continuous at $x = 3$.

Solution: Let $\epsilon > 0$ be given. Then

$$\begin{aligned} |f(x) - f(3)| &= |(3x - 4) - 5| \\ &= |3x - 9| \\ &= 3|x - 3| \\ &< \epsilon \text{ whenever } |x - 3| < \tfrac{\epsilon}{3}. \end{aligned}$$

We define $= \frac{\epsilon}{2}$. Then it follows that $\lim_{x\to 3} f(x) = f(3)$ and, hence f is continuous at $x = 3$.

(4) Show that $f(x) = x^3$ is continuous at $x = 2$.

Solution:
Since $f(2) = 8$, we need to prove that $\lim_{x\to 2} x^3 = 8 = 2^3$. Let $\epsilon > 0$ be given. Let us concentrate our attention on the open interval $(1,3)$ that contains $x = 2$ at its mid – point. Then

$$\begin{aligned} |f(x) - f(2)| &= |x^3 - 8| \\ &= |(x - 2)(x^2 + 2x + 4)| \\ &= |x - 2||x^2 + 2x + 4| \\ &\le |x - 2|(|x^2| + 2|x| + 4) \\ &\le |x - 2|(9 + 18 + 4) \\ &= 31|x - 2| \\ &< \epsilon \end{aligned}$$

provided $|x-2| < \frac{\epsilon}{31}$.

Since we are concentrating on the interval $(1,3)$ for which $|x-2| < 1$, we need to define δ to be the minimum of 1 and $\frac{\epsilon}{31}$. Thus, if we define $\delta = \min\left\{1, \frac{\epsilon}{31}\right\}$, then $|f(x) - f(2)| < \epsilon$ whenever $|x-2| < \delta$. By definition, $f(x)$ is continuous at $x = 2$.

(5) Show that $f(x) = \frac{1}{x}$ is continuous at every real number $c > 0$.

Solution: Let $\epsilon > 0$ be given. Let us concentrate on the interval $|x-c| \leq \frac{c}{2}$; that is $\frac{c}{2} \leq x \leq \frac{3c}{2}$. Clearly $x \neq 0$ in this interval. Then

$$
\begin{aligned}
|f(x) - f(c)| &= \left|\frac{1}{x} - \frac{1}{c}\right| \\
&= \left|\frac{c-x}{cx}\right| \\
&= |x-c| \cdot \frac{1}{c} \cdot \frac{1}{|x|} \\
&< |x-c| \cdot \frac{1}{c} \cdot \frac{2}{c} \\
&= \frac{2}{c^2}|x-c| \\
&< \epsilon \ \text{ whenever } |x-c| < \frac{c^2\epsilon}{2}.
\end{aligned}
$$

We define $\delta = \min\left\{\frac{c}{2}, \frac{c^2\epsilon}{2}\right\}$. Then for all x such that $|x-c| < \delta$, $\left|\frac{1}{x} - \frac{1}{c}\right| < \epsilon$. Hence, $\lim_{x\to c}\frac{1}{x} = \frac{1}{c}$ and the function $f(x) = \frac{1}{x}$ is continuous at each $c > 0$.

Exercise 1

1. Evaluate each of the following limits.

   (a) $\lim_{x\to 2}(x^4 - 3x^2 + 2x + 4)$ (b) $\lim_{x\to 1}\frac{x^2-1}{x^2-1}$ (c) $\lim_{x\to 0}\frac{\sin(2x)}{x}$ (d) $\lim_{x\to 0}\frac{\sin 5x}{\sin 7x}$

   (e) $\lim_{x\to 2}\frac{1}{x^2-4}$ (f) $\lim_{x\to 2}\frac{1}{x^2-4}$ (g) $\lim_{x\to 3}\frac{x^2-9}{x-3}$ (h) $\lim_{x\to 0}\frac{\sin 2x + \sin 3x}{x}$ (i) $\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}$

   (j) $\lim_{x\to\infty}\frac{3x^2+4x+2}{6x^6+3x^2+8}$

2. Suppose that a function f is defined and continuous on some open interval (a,b) and $a < c < b$. Prove that

   (i) If $f(c) > 0$, then there exists some $\delta > 0$ such that $f(x) > 0$ whenever $c - \delta < x < c + \delta$.

   (ii) If $f(c) < 0$, then there exists some $\delta > 0$ such that $f(x) < 0$ whenever $c - \delta < x < c + \delta$.

<u>Theorem 1.3.5</u> (Intermediate Value Theorem): Let $f:[a,b] \to \mathbb{R}$ be continuous. Suppose that $f(a) < f(b)$. Then for any v with $f(a) < v < f(b)$ there exists $c \in (a,b)$ such that $f(c) = v$.

<u>Example</u>: Let $f:[0,1] \to \mathbb{R}$ be given by $f(x) = x^2 + 6x + 1,\ x \in [0,1]$. Can we solve $f(x) = 2$?

Solution: The answer is yes. Since $f(0) = 1,\ f(1) = 8$ and $2 \in (1,8)$, there is a number $c \in [0,1]$ such that $f(c) = 2$ by the Intermediate Value Theorem.

§ 2.0 DIFFERENTIATION

We begin this section by reviewing the concept of differentiation for functions of one variable.

Definition 2.1.1: Let $f: D \subset \mathbb{R} \to \mathbb{R}$ and let x_0 be an interior point of D. (A point $x_0 \in D$ is an interior point of D means there is an $r > 0$ such that $(x_0 - r, x_0 + r) \subset D$.) Then f is differentiable at x_0 means there is a number, denoted by $f'(x_0)$ such that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

or equivalently, $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$

exists. The number $f'(x_0)$ is called the derivative of f at x_0.

Geometrically, the derivative of a function at x_0 is interpreted as the slope of the line tangent to the graph of f at the point $(x_0, f(x_0))$. Not every function is differentiable at every number in its domain even if that function is continuous and this is stated in the following theorem.

Theorem 2.1.2: If f is differentiable at c, then f is continuous at c. The converse is false.

Proof: Suppose that f is differentiable at c. Then

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$$

and $f'(c)$ is a real number. So,

$$\begin{aligned} \lim_{x \to c} f(x) &= \lim_{x \to c} \left[\left(\frac{f(x) - f(c)}{x - c} \right) (x - c) + f(c) \right] \\ &= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \cdot \lim_{x \to c} (x - c) + f(c) \\ &= f'(c) \cdot 0 + f(c) \\ &= f(c). \end{aligned}$$

Therefore, if f is differentiable at c, then f is continuous at c.

To prove that the converse is false, we consider the function $f(x) = |x|$. This function is continuous at $x = 0$. But

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \left[\frac{|x+h| - |x|}{h} \right] \\ &= \lim_{h \to 0} \frac{(|x+h| - |x|)(|x+h| + |x|)}{h(|x+h| + |x|)} \\ &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h(|x+h| + |x|)} \\ &= \lim_{h \to 0} \frac{2x + h}{|x+h||x|} \end{aligned}$$

$$= \frac{x}{|x|}$$

$$= \begin{cases} 1 & \text{for } x > 0 \\ -1 & \text{for } x < 0 \\ \text{undefined} & \text{for } x = 0 \end{cases}$$

Thus, $|x|$ is continuous at 0 but not differentiable at 0. ■

Theorem 2.1.3: Suppose that functions f and g are defined on some open interval (a, b) and $f'(x)$ and $g'(x)$ exist at each point $x \in (a, b)$. Then

(i) $(f + g)'(x) = f'(x) + g'(x)$ (The Sum Rule)

(ii) $(f - g)'(x) = f'(x) - g'(x)$ (The Difference Rule)

(iii) $(kf)'(x) = kf'(x)$, for each constant k. (The Multiple Rule)

(iv) $(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$ (The Product Rule)

(v) $\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$, if $g(x) \neq 0$ (The Quotient Rule)

Proof:

(i) $$(f + g)'(x) = \lim_{h\to 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x) + g'(x) .$$

(ii) $$(f - g)'(x) = \lim_{h\to 0} \frac{[f(x+h) - g(x+h)] - [f(x) - g(x)]}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} - \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x) - g'(x) .$$

(iii) $$(kf)'(x) = \lim_{h\to 0} \frac{kf(x+h) - kf(x)}{h}$$

$$= k \cdot \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= k \cdot f'(x) .$$

(iv) $$(f \cdot g)'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\big(f(x+h) - f(x)\big)g(x+h) + f(x)(g(x+h) - g(x))\right]$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \cdot \lim_{h\to 0} g(x+h) + f(x) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x)g(x) + f(x)g'(x) .$$

(v) $\left(\frac{f}{g}\right)'(x) = \lim_{h\to 0}\frac{1}{h}\left[\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}\right]$

$= \lim_{h\to 0}\frac{1}{h}\left[\frac{f(x+h)\cdot g(x) - g(x+h)f(x)}{g(x+h)g(x)}\right]$

$= \frac{1}{(g(x))^2}\lim_{h\to 0}\left[\frac{(f(x+h)-f(x))}{h}g(x) - f(x)\frac{(g(x+h)-g(x))}{h}\right]$

$= \frac{1}{(g(x))^2}\cdot\left[f'(x)g(x) - f(x)g'(x)\right]$

$= \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$, if $g(x) \neq 0$ ■

Remark

To emphasis the fact that the derivatives are taken with respect to the independent variable x, we use the following notation, as is customary:

$f'(x) = \frac{d}{dx}(f(x))$.

Based on Theorem 2.1.3 and definition of the derivative, we get the following theorem.

Theorem 2.1.4:

(i) $\frac{d}{dx}(k) = 0$, where k is a real constant

(ii) $\frac{d}{dx}(x^n) = nx^{n-1}$, for each real number x and natural number n.

(iii) $\frac{d}{dx}(\sin x) = \cos x$ for all real numbers (radian measure) x.

(iv) $\frac{d}{dx}(\cos x) = -\sin x$ ∀ real numbers (radian measure) x.

(v) $\frac{d}{dx}(\tan x) = \sec^2 x$ ∀ real numbers $x \neq (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}$.

(vi) $\frac{d}{dx}(\cot x) = -\csc^2 x$ ∀ real numbers $x \neq n\pi, n \in \mathbb{Z}$.

(vii) $\frac{d}{dx}(\sec x) = \sec x \tan x$ ∀ real numbers $x \neq (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}$.

(viii) $\frac{d}{dx}(\csc x) = -\csc x \cot x$ ∀ real numbers $x \neq n\pi, n \in \mathbb{Z}$.

Proof:

(i) $\frac{d}{dx}(k) = \lim_{h\to 0}\frac{k-k}{h}$

$= \lim_{h\to 0}\frac{0}{h}$

$= 0$

(ii) For each $n \in \mathbb{N}$, we get

$$\frac{d}{dx}(x^n) = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\left[x^n + nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \cdots + h^n - x^n\right] \quad \text{(Binomial exp.)}$$

$$= \lim_{h\to 0}\left[nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}h + \cdots + h^{n-1}\right]$$

$$= nx^{n-1}\,.$$

(iii) By definition, we get

$$\frac{d}{dx}(\sin x) = \lim_{h\to 0}\frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h}$$

$$= \lim_{h\to 0}\left[\cos x\frac{\sin h}{h} - \sin x\left(\frac{1-\cos h}{h}\right)\right]$$

$$= \cos x\cdot 1 - \sin x\cdot 0$$

$$= \cos x$$

since $\lim_{h\to 0}\frac{\sin h}{h} = 1,\quad \lim_{h\to 0}\frac{1-\cos h}{h} = 0.$

(iv) $\frac{d}{dx}(\cos x) = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$

$$= \lim_{h\to 0}\frac{1}{h}[\cos x\cos h - \sin x\sin h - \cos x]$$

$$= \lim_{h\to 0}\left[-\sin x\cdot\frac{\sin h}{h} - \cos x\left(\frac{1-\cos h}{h}\right)\right]$$

$$= -\sin x\cdot 1 - \cos x\cdot 0$$

$$= -\sin x\,.$$

(v) $\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$

$$= \frac{\cos x(\sin x)' - \sin x(\cos x)'}{(\cos x)^2}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x,\ x \neq (2n+1)\frac{\pi}{2},\ n \in \mathbb{Z}.$$

(vi) Using the quotient rule and part (iii) and (iv), we get

$$\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$$

$$= \frac{(\sin x)(\cos x)' - (\cos x)(\sin x)'}{(\sin x)^2}$$

$$= \frac{-\sin^2 x - \cos^2 x}{(\sin x)^2}$$

$$= \frac{-1}{(\sin x)^2}$$

$$= -\csc^2 x \ , x \neq n\pi, n \in \mathbb{Z}.$$

(vii) $\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right)$

$$= \frac{(\cos x)\cdot 0 - 1\cdot(\cos x)'}{(\cos x)^2}$$

$$= \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x}$$

$$= \sec x \tan x \ , \ x \neq (2n+1)\frac{\pi}{2}, \ n \in \mathbb{Z}.$$

(viii) $\frac{d}{dx}(\csc x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right)$

$$= \frac{\sin x\cdot 0 - 1\cdot(\sin x)'}{(\sin x)^2}$$

$$= \frac{1}{\sin x}\cdot\frac{-\cos x}{\sin x}$$

$$= -\csc x \cot x \ , x \neq n\pi, n \in \mathbb{Z} \qquad \blacksquare$$

2.1.5 Examples

1. (a) Show that $f(x) = x^2$ is differentiable everywhere.

Solution: $\lim_{x\to x_0}\frac{x^2 - {x_0}^2}{x - x_0} = \lim_{x\to x_0}(x + x_0) = 2x_0$. Hence $f'(x_0)$ exists and equals $2x_0$.

(b) Show that $f: \mathbb{R}\backslash\{0\} \to \mathbb{R}$, $f(x) = \frac{1}{x}$ is differentiable at $x_0 \neq 0$.

Solution: $\lim_{x\to x_0}\frac{f(x) - f(x_0)}{x - x_0} = \lim_{x\to x_0}\frac{\frac{1}{x} - \frac{1}{x_0}}{x - x_0}$

$$= \lim_{x\to x_0}\frac{\frac{x_0 - x}{x_0 x}}{x - x_0}$$

$$= \lim_{x\to x_0} -\frac{1}{x_0 x}$$

$$= -\frac{1}{{x_0}^2}$$

In the last step, we use the fact that $\frac{1}{x}$ is continuous at $x_0 \neq 0$.

(c) The function $f(x) = |x|$ is not differentiable at $h = 0$.

Solution: $\lim_{h\to 0^+}\frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^+}\frac{|h|}{h} = \lim_{h\to 0^+}\frac{h}{h} = 1$,

$$\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-}\frac{|h|}{h} = \lim_{h\to 0^+}\frac{-h}{h} = -1.$$

Hence, $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$ does not exist and f is not differentiable at 0.

2. Compute the following derivatives:

(i) $\frac{d}{dx}(5x^3 - 3x^2 + 2x + 10)$ (ii) $\frac{d}{dx}(4\sin x - 3\cos x)$ (iii) $\frac{d}{dx}(x\sin x + x^2\cos x)$

(iv) $\frac{d}{dx}\left(\frac{x^3+1}{x^2+4}\right)$

Solution:

(i) $$\frac{d}{dx}(5x^3 - 3x^2 + 2x + 10) = 5\frac{d}{dx}(x^3) - 3\frac{d}{dx}(x^2) + 2\frac{d}{dx}(x) + 0$$
$$= 15x^2 - 6x + 2.$$

(ii) $$\frac{d}{dx}(4\sin x - 3\cos x) = 4\frac{d}{dx}(\sin x) - 3\frac{d}{dx}(\cos x)$$
$$= 4\cos x - 3(-\sin x)$$
$$= 4\cos x + 3\sin x\ .$$

(iii) Using the sum and product rules, we get

$$\frac{d}{dx}(x\sin x + x^2\cos x) = \frac{d}{dx}(x\sin x) + \frac{d}{dx}(x^2\cos x)$$
$$= \left[\frac{d}{dx}(x)\sin x + x\frac{d}{dx}(\sin x)\right] + \left[\frac{d}{dx}(x^2)\cos x + x^2\frac{d}{dx}(\cos x)\right]$$
$$= 1\sin x + x\cos x + 2x\cos x + x^2(-\sin x)$$
$$= \sin x + 3x\cos x - x^2\sin x\ .$$

(iv) $$\frac{d}{dx}\left(\frac{x^3+1}{x^2+4}\right) = \frac{(x^2+4)\frac{d}{dx}(x^3+1) - (x^3+1)\frac{d}{dx}(x^2+4)}{(x^2+4)^2}$$
$$= \frac{(x^2+4)(3x^2) - (x^3+1)(2x)}{(x^2+4)^2}$$
$$= \frac{3x^4 - 2x^3 + 12x^2 - 2x}{(x^2+4)^2}\ .$$

Exercise 2

1. From the definition, (a) prove that $\frac{d}{dx}(x) = 1$

   (b) prove that $\frac{d}{dx}(x^3) = 3x^2$

2. Compute the derivative $\frac{d}{dx}(3\cot x + 5\csc x + 20)$

2.2 The Chain Rule

Suppose we have two functions, u and y, related by the equations: $u = g(x)$ and $y = f(u)$. Then $y = (f \circ g)(x) = f(g(x))$. The chain rule deals with the derivative of the composition and may be stated as the following theorem.

Theorem 2.2.1(The Chain Rule): Suppose that g is defined in an open interval I containing c, and f is defined in an open interval J containing $g(c)$, such that $g(x)$ is in J for all $x \in I$. If g is differentiable at c, and f is differentiable at $g(c)$, then the composition $(f \circ g)$ is differentiable at c and $(f \circ g)'(x) = f'(g(c)) \cdot g'(c)$. In general, if $u = g(x)$ and $y = f(u)$, then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

Proof: Let F be defined on J such that

$$F(u) = \begin{cases} \frac{f(u)-f(g(c))}{u-g(c)} & \text{if } u \neq g(c) \\ f'(g(c)) & \text{if } u = g(c) \end{cases}$$

since f is differentiable at $g(c)$

$$\begin{aligned} \lim_{u \to g(c)} F(u) &= \lim_{u \to g(c)} \frac{f(u)-f(g(c))}{u-g(c)} \\ &= f'(g(c)) \\ &= F(g(c)) \end{aligned}$$

Therefore, F is continuous at $g(c)$. By the definition of F, $f(u) - f\big(g(c)\big) = F(u)(u - g(c)$ for all $u \in J$. For each $x \in I$, we let $y = g(x)$ on I. Then

$$\begin{aligned} (f \circ g)'(c) &= \lim_{x \to c} \frac{(f\circ g)(x)-(f\circ g)(c)}{x-c} \\ &= \lim_{x \to c} \frac{f(g(x))-f(g(c))}{g(x)-g(c)} \cdot \frac{g(x)-g(c)}{x-c} \\ &= \lim_{u \to g(c)} F(u) \cdot \lim_{x \to c} \frac{g(x)-g(c)}{x-c} \\ &= f'(g(c)) \cdot g'(c) \end{aligned}$$

It follows that $f \circ g$ is differentiable at c. The general result follows by replacing c by the independent variable x. This completes the proof.

2.2.2 Examples

1. Let $y = u^2 + 1$ and $u = x^3 + 4$. Then $\frac{dy}{dx} = 2u$ and $\frac{du}{dx} = 3x^2$. Therefore,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 2u \cdot 3x^2 = 6x^2(x^3 + 4).$$

Using the composition notation, we get

$$y = (x^3 + 4)^2 + 1 = x^6 + 8x^3 + 17$$

and $\frac{dy}{dx} = 6x^5 + 24x^2 = 6x^2(x^3 + 4)$.

Using $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$, we see that $(f \circ g)(x) = (x^3 + 4)^2 + 1$ and

$$\begin{aligned}(f \circ g)'(x) &= f'(g(x)) \cdot g'(x)\\ &= 2(x^3 + 4) \cdot (3x^2)\\ &= 6x^2(x^3 + 4).\end{aligned}$$

2. Suppose that $y = \sin(x^2 + 3)$.

We let $u = x^2 + 3$, and $y = \sin u$. then

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx}\\ &= (\cos u)(2x)\\ &= (\cos(x^2 + 3)) \cdot (2x).\end{aligned}$$

PP: Evaluate $\frac{dy}{dx}$ if $y = (\cos(3x + 1))^5$.

2.3 Differentiation of Inverse Functions

One of the applications of chain rule is to compute the derivatives of inverse functions.

<u>Theorem 2.3.1</u>: Suppose that a function f has an inverse, f^{-1}, on an open interval I. If $u = f^{-1}(x)$ then

(i) $\frac{du}{dx} = \frac{1}{\left(\frac{dx}{du}\right)}$

(ii) $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(u)}$

<u>Proof</u>: By comparison, $x = f(f^{-1}(x)) = x$. Hence, by the chain rule

$$1 = \frac{dx}{dx} = f'\left(f^{-1}(x)\right) \cdot (f^{-1})'(x)$$

and $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$. In the $u = f^{-1}(x)$ notation, we have $\frac{du}{dx} = \frac{1}{\left(\frac{dx}{du}\right)}$.

2.3.2 Examples

(1) Let $u = \arcsin x$, $-1 \le x \le 1$ and $-\frac{\pi}{2} \le u \le \frac{\pi}{2}$. Then $x = \sin u$ and by the chain rule, we get

$$\begin{aligned}1 = \frac{dx}{dx} &= \frac{d(\sin u)}{du} \cdot \frac{du}{dx}\\ &= \cos u \cdot \frac{du}{dx}\end{aligned}$$

$$\Rightarrow \frac{du}{dx} = \frac{1}{\cos u}$$

Therefore, $\frac{d}{dx}(\arcsin x) = \frac{1}{\cos u}, -\frac{\pi}{2} \le u \le \frac{\pi}{2},$

$$= \frac{1}{\sqrt{1-\sin^2 u}}$$

$$= \frac{1}{\sqrt{1-x^2}}, -1 < x < 1$$

Thus, $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}, -1 < x < 1$. We note that $x = \pm 1$ are excluded.

(2) Let $u = \operatorname{arcsec} x,\ x \in (-\infty, -1] \cup [1, \infty)$ and $u \in [0, \frac{\pi}{2}) \cup \left(\frac{\pi}{2}, \pi\right)$. Then, $x = \sec u$

$$1 = \frac{dx}{dx} = \sec u \tan u \cdot \frac{du}{dx},\ \ u \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$$

$$\Rightarrow \frac{du}{dx} = \frac{1}{\sec u \tan u}, u \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$$

$$= \frac{1}{|\sec u|\sqrt{\sec^2 u - 1}}$$

$$= \frac{1}{|x|\sqrt{x^2-1}}, x \in (-\infty, -1) \cup (1, \infty).$$

Thus, $\frac{d}{dx}(\operatorname{arcsec} x) = \frac{1}{|x|\sqrt{x^2-1}}, x \in (-\infty, -1) \cup (1, \infty)$.

Theorem 2.3.3 (The Inverse Trigonometric function): The following differentiation formulae are valid for the inverse trigonometric functions:

(i) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, -1 < x < 1.$

(ii) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}, -1 < x < 1.$

(iii) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}, -\infty < x < \infty$

(iv) $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}, -\infty < x < \infty$

(v) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2+1}}, -\infty < x < -1\ or\ 1 < x < \infty$

(vi) $\frac{d}{dx}(\csc^{-1} x) = \frac{-1}{|x|\sqrt{x^2+1}}, -\infty < x < -1\ or\ 1 < x < \infty.$

Theorem 2.3.4 (Logarithmic and Exponential functions)

(i) $\frac{d}{dx}(\ln x) = \frac{1}{x}$ for all $x > 0$ [Note: $\ln x = \log_e x$]

(ii) $\frac{d}{dx}(e^x) = e^x$ for all real x

(iii) $\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$ for all $x > 0$ and $b \neq 1$

(iv) $\frac{d}{dx}(b^x) = b^x(\ln b)$ for all real $x, b > 0$ and $b \neq 1$

(v) $\frac{d}{dx}(u(x))^{v(x)} = (u(x))^{v(x)}\left[v'(x)\ln(u(x)) + v(x)\frac{u'(x)}{u(x)}\right].$

Proof: (i) and (ii) left as exercise.

(iii) By definition, for all $x > 0, b > 0$ and $b \neq 1,\ \log_b x = \frac{\ln x}{\ln b}$. Then

$$\frac{d}{dx}(\log_b x) = \frac{d}{dx}\left(\left(\frac{1}{\ln b}\right)\ln x\right)$$
$$= \left(\frac{1}{\ln b}\right)\cdot\frac{1}{x}$$
$$= \frac{1}{x\ln b}\,.$$

(iv) By definition, for real $> 0, b > 0$ and $b \neq 1$, $b^x = e^{x\ln b}$. Therefore,

$$\frac{d}{dx}(b^x) = \frac{d}{dx}\left(e^{x\ln b}\right)$$
$$= e^{x\ln b}\cdot\frac{d}{dx}(x\ln b) \quad \text{(by applying the chain rule)}$$
$$= b^x\ln b\,.$$

(v) $$\frac{d}{dx}(u(x))^{v(x)} = \frac{d}{dx}\left\{e^{v(x)\ln(u(x))}\right\}$$
$$= e^{v(x)\ln(u(x))}\left\{v'(x)\ln\big(u(x)\big) + v(x)\frac{u'(x)}{u(x)}\right\}$$
$$= (u(x))^{v(x)}\left\{v'(x)\ln\big(u(x)\big) + v(x)\frac{u'(x)}{u(x)}\right\}.$$

Example

1. Let $y = \log_{10}(x^2+1)$. Then

$$\frac{d}{dx}(\log_{10}(x^2+1)) = \frac{d}{dx}\left(\frac{\ln(x^2+1)}{\ln 10}\right)$$
$$= \frac{1}{\ln 10}\left(\frac{1}{x^2+1}\cdot 2x\right)$$
$$= \frac{2x}{(x^2+1)\ln 10}$$

2. Let $y = e^{x^2+1}$. Then, by the chain rule, we get

$$\frac{dy}{dx} = e^{x^2+1}\cdot 2x$$
$$= 2xe^{x^2+1}\,.$$

3. Let $y = 10^{(x^3+2x+1)}$. By definition and the chain rule, we get

$$\frac{dy}{dx} = 10^{(x^3+2x+1)}\cdot(\ln 10)\cdot(3x^2+2).$$

Theorem 2.3.5 (Differentiation of Hyperbolic functions)

(i) $\frac{d}{dx}(\sinh x) = \cosh x$

(ii) $\frac{d}{dx}(\cosh x) = \sinh x$

(iii) $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

(iv) $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

(v) $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$

(vi) $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$.

Proof:

(i)
$$\begin{aligned}\frac{d}{dx}(\sinh x) &= \frac{d}{dx}\left(\frac{1}{2}(e^x - e^{-x})\right)\\ &= \frac{1}{2}\left(e^x - e^{-x}(-1)\right)\\ &= \frac{1}{2}(e^x + e^{-x})\\ &= \cosh x \ .\end{aligned}$$

(ii)
$$\begin{aligned}\frac{d}{dx}(\cosh x) &= \frac{d}{dx}\left(\frac{1}{2}(e^x + e^{-x})\right)\\ &= \frac{1}{2}\left(e^x + e^{-x}(-1)\right)\\ &= \frac{1}{2}(e^x - e^{-x})\\ &= \sinh x \ .\end{aligned}$$

(iii)
$$\begin{aligned}\frac{d}{dx}(\tanh x) &= \frac{d}{dx}\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)\\ &= \frac{(e^x+e^{-x})(e^x+e^{-x})-(e^x-e^{-x})(e^x-e^{-x})}{(e^x+e^{-x})^2}\\ &= \frac{4}{(e^x+e^{-x})^2}\\ &= \left(\frac{2}{e^x+e^{-x}}\right)^2\\ &= \sinh^2 x \ .\end{aligned}$$

(iv)
$$\begin{aligned}\frac{d}{dx}(\coth x) &= \frac{d}{dx}\left(\frac{e^x + e^{-x}}{e^x - e^{-x}}\right), x \neq 0\\ &= \frac{(e^x-e^{-x})(e^x-e^{-x})-(e^x+e^{-x})(e^x+e^{-x})}{(e^x-e^{-x})^2}, \quad x \neq 0\end{aligned}$$

$$= \frac{-4}{(e^x - e^{-x})^2}, \quad x \neq 0$$

$$= -\left(\frac{2}{e^x - e^{-x}}\right)^2, \quad x \neq 0$$

$$= -\operatorname{csch}^2 x, \quad x \neq 0.$$

(v) $$\frac{d}{dx}(\operatorname{sech} x) = \frac{d}{dx}\left(\frac{2}{e^x + e^{-x}}\right)$$

$$= \frac{(e^x + e^{-x}) \cdot 0 - 2(e^x - e^{-x})}{(e^x + e^{-x})^2}$$

$$= \frac{-2}{e^x + e^{-x}} \cdot \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$= -\operatorname{sech} x \tanh x.$$

(vi) $$\frac{d}{dx}(\operatorname{csch} x) = \frac{d}{dx}\left(\frac{2}{e^x - e^{-x}}\right)$$

$$= \frac{(e^x - e^{-x}) \cdot 0 - 2(e^x + e^{-x})}{(e^x - e^{-x})^2}, \quad x \neq 0$$

$$= \frac{-2}{e^x + e^{-x}} \cdot \frac{e^x + e^{-x}}{e^x - e^{-x}}, \quad x \neq 0$$

$$= -\operatorname{csch} x \coth x, \quad x \neq 0.$$

Theorem 2.3.6 (Inverse Hyperbolic Functions)

(i) $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$

(ii) $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}, \quad x > 1$

(iii) $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}, \quad |x| < 1$

Proof: Exercise.

2.4 Implicit Differentiation

In an application, two variables can be related by an equation such as (i) $x^2 + y^2 = 20$ (ii) $x^3 + y^3 = 4xy$ (iii) $\sin y + \cos 3y = \sin 2y$. In such cases, it is not always practical or desirable to solve for one variable explicitly in terms of the other to compute derivative. Instead, we may implicitly assume that y is some function of x and differentiable each term of the equation with respect to x. Then we solve for y' noting any conditions under which the derivative may or may not exist.

Examples

(1) Find $\frac{dy}{dx}$ if $x^2 + y^2 = 20$.

Solution: Assume that y is to be considered as a function of x, we differentiate each term of the equation with respect to x.

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(20)$$

$$2x + 2y\left(\frac{dy}{dx}\right) = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}\ ,\ \text{provided}\ \ y \neq 0.$$

(2) Compute $\frac{dy}{dx}$ for the equation $x \sin y + \cos 3y = \sin 2y$.

Solution: $\frac{d}{dx}(x \sin y) + \frac{d}{dx}(\cos 3y) = \frac{d}{dx}(\sin 2y)$

$$\left[\left(\frac{dx}{dx}\right)(\sin y) + x\left(\frac{d}{dx}(\sin y)\right)\right] + (-3\sin 3y)\frac{dy}{dx} = (\cos 2y)\left(2\frac{dy}{dx}\right)$$

$$\Rightarrow \sin y + x(\cos y)\frac{dy}{dx} - 3\sin(3y)\frac{dy}{dx} = -\sin y$$

$$\Rightarrow [x\cos y - 3\sin 3y - 2\cos 2y]\frac{dy}{dx} = -\sin y$$

$\Rightarrow \frac{dy}{dx} = -\frac{\sin y}{x\cos y - 3\sin 3y - 2\cos 2y}$ whenever $x\cos y - 3\sin 3y - 2\cos 2y \neq 0.$

Using the definition: Let $f: D \subset \mathbb{R} \to \mathbb{R}$ and let a be an interior point of D, then f is differentiable at a means there is a number, $f'(a)$, such that

$$\lim_{x\to a}\frac{f(x) - f(a) - (x-a)f'(a)}{|x-a|} = 0,$$

We now give the ***definition of differentiability for functions of several variables*** as follows:

Definition 2.5.1: Let $f: D \subset \mathbb{R}^n \to \mathbb{R}$ and let P_0 be an interior point of D. (A point $P_0 \in D$ is an interior point of D mean there is an $r > 0$ such that $\{P \in \mathbb{R}^n;\ |P - P_0 < r|\} \subset D.$) Then f is differentiable at P_0 means there is a vector, denoted by $f'(P_0)$ for now, such that

$$\lim_{P\to P_0}\frac{f(P) - f(P_0) - \overrightarrow{(P_0P)}f'(P_0)}{|P - P_0|} = 0.$$

For functions of two variables, the definition becomes the following.

Definition 2.5.2: Let $f: D \subset \mathbb{R}^2 \to \mathbb{R}$ and let (x_0, y_0) be an interior point of D. Then f is differentiable at (x_0, y_0) means there are two numbers $f_1(x_0, y_0)$ and $f_2(x_0, y_0)$ such that

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)-f(x_0,y_0)-(x-x_0)f_1(x_0,y_0)-(y-y_0)f_2(x_0,y_0)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0 .$$

Here, we are dealing with partial derivatives and we used to denote partial derivatives as

$$f_1(x,y) = \frac{\partial f}{\partial x}(x,y) = \frac{\partial}{\partial x} f(x,y) = f_x(x,y).$$

For example: Let $(x,y) = \sqrt{y^2 - x^2} = (y^2 - x^2)^{1/2}$. Then

$$f_1(x,y) = -x(y^2 - x^2)^{-1/2} \text{ and } f_2(x,y) = y(y^2 - x^2)^{-1/2} .$$

Implicit function

For a given function $f(x,y)$ with $f = 0$ and $\frac{\partial f}{\partial x} \neq 0$ at the point (x_0, y_0), there corresponds a unique function $y(x)$ in the neighbourhood of (x_0, y_0).

Let us consider the equation

$$f(x,y,u,v) = 0 \qquad (1)$$

$$g(x,y,u,v) = 0 \qquad (2)$$

Under certain circumstances, we can unravel equations (1) and (2), either algebraically or numerically, to form $u = u(x,y),\ v = v(x,y)$. The conditions for the existence of such a functional dependency can be bound by differentiation of the original equations; for example, differentiating equation (1) gives

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv = 0 \qquad (3)$$

Holding y constant and dividing by dx, we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = 0 \qquad (4)$$

Operating on equation (2) in the same manner, we get

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial x} = 0 \qquad (5)$$

Similarly, holding x constant and dividing by dy, we get

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = 0 \qquad (6)$$

$$\frac{\partial g}{\partial y} + \frac{\partial g}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial y} = 0 \qquad (7)$$

Equations (4) and (5) can be solved for $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$, and equations (6) and (7) can be solved for $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ by using the well known Crammer's rule. To solve for $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$, we first write equation (4) and (5) in matrix form:

$$\begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \end{pmatrix} = \begin{pmatrix} -\frac{\partial f}{\partial x} \\ -\frac{\partial g}{\partial x} \end{pmatrix} \qquad (8)$$

Thus, from Cramer's rule, we have

$$\frac{\partial u}{\partial x} = \frac{\begin{vmatrix} -\frac{\partial f}{\partial x} & \frac{\partial f}{\partial v} \\ -\frac{\partial g}{\partial x} & \frac{\partial g}{\partial v} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{vmatrix}} = -\frac{\frac{\partial(f,g)}{\partial(x,v)}}{\frac{\partial(f,g)}{\partial(u,v)}}; \qquad \frac{\partial v}{\partial x} = \frac{\begin{vmatrix} \frac{\partial f}{\partial u} & -\frac{\partial f}{\partial x} \\ \frac{\partial g}{\partial u} & -\frac{\partial g}{\partial x} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{vmatrix}} = -\frac{\frac{\partial(f,g)}{\partial(u,x)}}{\frac{\partial(f,g)}{\partial(u,v)}}. \qquad (9)$$

In a similar fashion, we can form expressions for $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$:

$$\frac{\partial u}{\partial y} = \frac{\begin{vmatrix} -\frac{\partial f}{\partial y} & \frac{\partial f}{\partial v} \\ -\frac{\partial g}{\partial y} & \frac{\partial g}{\partial v} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{vmatrix}} = -\frac{\frac{\partial(f,g)}{\partial(y,v)}}{\frac{\partial(f,g)}{\partial(u,v)}}; \qquad \frac{\partial v}{\partial y} = \frac{\begin{vmatrix} \frac{\partial f}{\partial u} & -\frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial u} & -\frac{\partial g}{\partial y} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{vmatrix}} = -\frac{\frac{\partial(f,g)}{\partial(u,y)}}{\frac{\partial(f,g)}{\partial(u,v)}}. \qquad (10)$$

Here we take the **Jacobian matrix** J of the transformation to be defined as

$$J = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{gf}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix}. \qquad (11)$$

This is distinguished from the **Jacobian determinant** $|J|$, defined as

$$|J| = det = \frac{\partial(f,g)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{vmatrix}. \qquad (12)$$

If $|J| \neq 0$, the derivatives exist, and we indeed can form $u(x,y)$ and $v(x,y)$. *This is the condition for existence of implicit function conversion.*

2.5.4 Example

If $\quad x + y + u^4 + u + v = 0$ ....(i)

$xy + uv = 1$ ....(ii) evaluate $\frac{\partial u}{\partial x}$.

Solution: Here we have four unknowns in two equations. In principle, we could solve for $u(x,y)$ and $v(x,y)$ and then determine all partial derivatives, such as the one desire. In practice, this is not always possible; for example, there is no general solution to sixth order polynomial equation as in the case of quadratic equation. So we need to use the method discussed above to be able to provide the desire solution.

Equations (i) and (ii) are rewritten as

$f(x,y,u,v) = x + y + u^4 + u + v = 0$ ....(iii)

$$g(x,y,u,v) = xy + uv - 1 = 0 \qquad \text{....................(iv)}$$

Using the formula from equation (9) to solve for the desired derivative, we get

$$\frac{\partial u}{\partial x} = \frac{\begin{vmatrix} -\frac{\partial f}{\partial x} & \frac{\partial f}{\partial v} \\ -\frac{\partial g}{\partial x} & \frac{\partial g}{\partial v} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{vmatrix}} \qquad \text{......................(v)}$$

Substituting, we get

$$\frac{\partial u}{\partial x} = \frac{\begin{vmatrix} -1 & 1 \\ -y & u \end{vmatrix}}{\begin{vmatrix} 4u^3+1 & 1 \\ v & u \end{vmatrix}} = \frac{y-u}{u(4u^3+1)-v}, \qquad v \neq u(4u^3+1) \qquad \text{...............(vi).}$$

Note: When $v = u(4u^3 + 1)$, that is, when the relevant Jacobian determinant is zero; at such points, we can neither determine $\frac{\partial u}{\partial x}$ nor $\frac{\partial u}{\partial y}$. Thus, for such points we can not form $u(x,y)$.

2.5.5 Functional dependence

Let $u = u(x,y)$ and $v = v(x,y)$. If we can write $u = g(v)$ or $v = h(u)$, then u and v are said to be **<u>functionally dependent</u>**, otherwise, **<u>functionally independent</u>**. If functionally dependence between u and v exists, then we can consider $f(u,v) = 0$. So,

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = 0 \qquad (13)$$

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = 0 \qquad (14)$$

In matrix form, this is

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial u} \\ \frac{\partial f}{\partial v} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since the right hand side is zero, and we desire a non-trivial solution, the determinant of the coefficient matrix must be zero for <u>functional dependency</u> i.e.

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0.$$

Note, since $\det J = \det J^T$, that this is equivalent to

$$|J| = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial(u,v)}{\partial(x,y)} = 0$$

That is, Jacobian determinant must be zero for functional dependence and $|J| \neq 0$ for functional independence.

Examples

(1) Determine if $u = y + z$ (i)

$v = x + 2z^2$ (ii)

$w = x - 4yz - 2y^2$ (iii)

are functionally dependent or functionally independent.

Solution: The determinant of the resulting coefficient matrix by extension to 3 functions of three variables is

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{vmatrix} \quad \text{(iv)}$$

$$= \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & -4z - 4y \\ 1 & 4z & -4y \end{vmatrix} \quad \text{(v)}$$

$$= (-1)(-4y - (-4)(y + z)) + (1)(4z) \quad \text{(vi)}$$

$$= 4y - 4y - 4z + 4z \quad \text{(vii)}$$

$$= 0 \quad \text{(viii)}$$

So, u, v, w are functionally dependent.

In fact, $w = v - 2u^2$.

(2) Given that $f = e^{xy} + 2x$, $g = xy + y^2 + \sin y$. Determine whether f and g are functionally dependent or not.

Solution: Exercise

2.5.6 Maxima and minima

Consider the real valued function $f(x)$, where $x \in [a, b]$. Extrema are at $x = x_m$, where $f'(x_m) = 0$, if $x_m \in [a, b]$. It is a local minimum, a local maximum, or an inflection point according to whether $f''(x_m)$ positive, negative is or zero, respectively.

Now consider a function of two variables $f(x, y)$, with $x \in [a, b], y \in [c, d]$. A necessary condition for an extremum is

$$\frac{\partial f}{\partial x}(x_m, y_m) = \frac{\partial f}{\partial y}(x_m, y_m) = 0 \quad (17)$$

where $x_m \in [a, b]$, $y_m \in [c, d]$. Next we find the **Hessian matrix**:

$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} \quad (18)$$

We use H and its element to determine the character of the local extremum:

(i) f is a maximum if $\frac{\partial^2 f}{\partial x^2} < 0, \frac{\partial^2 f}{\partial y^2} < 0$, and $\frac{\partial^2 f}{\partial x \partial y} < \sqrt{\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right)}$

(ii) f is a minimum if $\frac{\partial^2 f}{\partial x^2} > 0, \frac{\partial^2 f}{\partial y^2} > 0$, and $\frac{\partial^2 f}{\partial x \partial y} < \sqrt{\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right)}$

(iii) f is a saddle otherwise, as long as $\det H \neq 0$, and

(iv) If $\det H = 0$, higher order terms need to be considered.

Examples

(1) Consider extrema of $f = x^2 - y^2$.

Solution: Equating partial derivatives with respect to x and y to zero, we get

$$\frac{\partial f}{\partial x} = 2x = 0 \qquad \text{(i)}$$

$$\frac{\partial f}{\partial y} = -2y = 0 \qquad \text{(ii)}$$

This gives $x = 0$, $y = 0$. For these values we find that

$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}. \qquad \text{(iii)}$$

Since $\det H = -4 \neq 0$, and $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2}$ have different signs, the equilibrium is a saddle point.

(2) Find the local maximum, local minimum and saddle points (if any) of $f(x, y) = x^4 + y^4 - 4xy + 1$.

<u>Solution</u>: First $f_x = 4x^3 - 4y$ and $f_y = 4y^3 - 4x$. Now we proceed to solve $4x^3 - 4y = 0$ and $4y^3 - 4x = 0$ for the critical points. The two equations are equivalent to $y = x^3$ and $x = y^3$. Substituting one into the other, we obtain $x^9 - x = 0$. That is $x(x+1)(x-1)(x^2+1)(x^4+1) = 0$. Thus the real solutions are $x = 0, -1, 1$. Therefore, the critical points are $(0,0), (-1,-1)$, and $(1,1)$. To apply the second derivative test, we compute the second order partial derivatives.

$f_{xx} = 12x^2$, . $f_{yy} = -4$. Thus $D(x, y) = f_{xx} f_{yy} - {f^2}_{xy} = 144x^2y^2 - 16$.

At $(0,0)$, $D(0,0) = -16 < 0$. Hence, f has a saddle point at $(0,0)$. At $(-1,-1)$, $D(-1,-1) = 128 > 0$ and $f_{xx}(-1,-1) = 12 > 0$. Hence, f has a local minimum at $(-1,-1)$. At $(1,1)$, $D(1,1) = 128 > 0$ and $f_{xx}(1,1) = 12 > 0$. Hence, f has a local minimum at $(1,1)$. ■

2.5.7 Lagrange's Multiplier

Lagrange multipliers- Simplest case

Consider a function f of just two variables x and y. Say we want to find a stationary of the form $f(x,y)$ subject to a single constraint of the form $g(x,y)=0$.

(i) Introduce a single new variable λ- we call λ a Lagrange multiplier.

(ii) Find all sets of values of (x,y,λ) such that $\nabla f=\lambda\nabla g$ and $g(x,y)=0$ where $\nabla f=\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)$ i.e. $\frac{\partial f}{\partial x}=\lambda\frac{\partial g}{\partial x}$ and $\frac{\partial f}{\partial y}=\lambda\frac{\partial g}{\partial y}$ and $g(x,y)=0$.

(iii) Evaluate $f(x,y)$ at each of these points. We can often identify the largest/smallest value as the maximum/minimum of $f(x,y)$ subject to the constraint, taking account of whether f is bounded or unbounded above/below.

Example

1. Maximize $f(x,y)=xy$ subject to $x+y=1$ i.e. subject to $g(x,y)=x+y-1$.
   Solution: Since we have one constraint and so we introduce one Lagrange multiplier λ. Compute $\frac{\partial f}{\partial x}=y$, $\frac{\partial f}{\partial y}=x$, $\frac{\partial g}{\partial x}=1$, $\frac{\partial g}{\partial y}=1$ and solve the (two + one) equations

$$\frac{\partial f}{\partial x}=\lambda\frac{\partial g}{\partial x} \quad \text{i.e.} \quad y=\lambda \quad \text{(i)}$$

$$\frac{\partial f}{\partial y}=\lambda\frac{\partial g}{\partial y} \quad \text{i.e.} \quad x=\lambda \quad \text{(ii)}$$

$$g(x,y)=0 \quad \text{i.e.} \quad x+y=1 \quad \text{(iii)}$$

   Substituting (i) and (ii) in (iii) gives $2\lambda=1$ i.e. $\lambda=\frac{1}{2}$, so from (i) and (ii) the function has a stationary point subject to the constraint (here a maximum), at $x=\frac{1}{2}$, $y=\frac{1}{2}$.

2. Find the extreme value of $f(x,y)=x^2+2y^2$ on the circle $x^2+y^2=1$.
   Solution: $f(x,y)$ is subject to $g(x,y)=0$ where $g(x,y)=x^2+y^2-1$. So we introduce one Lagrange multiplier λ. Compute $\frac{\partial f}{\partial x}=2x$, $\frac{\partial f}{\partial y}=4y$, $\frac{\partial g}{\partial x}=2x$, $\frac{\partial g}{\partial y}=2y$ and solve the (two + one) equations

$$\frac{\partial f}{\partial x}=\lambda\frac{\partial g}{\partial x} \quad \text{i.e.} \quad 2x=\lambda 2x \quad \text{(i)}$$

$$\frac{\partial f}{\partial y}=\lambda\frac{\partial g}{\partial y} \quad \text{i.e.} \quad 4y=\lambda 2y \quad \text{(ii)}$$

$$g(x,y)=0 \quad \text{i.e.} \quad x^2+y^2=1 \quad \text{(iii)}$$

Equation (i) ⇒ $\lambda = 1$ or $x = 0$; (ii) ⇒ either $\lambda = 2$ or $y = 0$. So possible solutions are $x = 0, \lambda = 2, y = \pm 1$ and $y = 0, \lambda = 1, x = \pm 1$ where $f(0, \pm 1) = 2$ (max), while $f(\pm 1, 0) = 1$ (min).

Lagrange multipliers- General number of variables and constraints

The method easily generalises to finding the stationary points of a function f with n variables subject to k independent constraints. E.g. consider a function $f(x, y, z)$ of three variables x, y, z subject to two constraints $g(x, y, z) = 0$ and $h(x, y, z) = 0$, then:

(i) at a stationary point ∇f is the plane determined by ∇g and ∇h

(ii) introduce two Lagrange multipliers, say λ and μ

(iii) find all sets of values x, y, z, λ, μ satisfying the five (i.e. 3+2) equations $\nabla f = \lambda \nabla g + \mu \nabla h$ and $g(x, y, z) = 0$ and $h(x, y, z) = 0$.

Again consider the general case of finding a stationary point of a function $f(x_1, \ldots, x_n)$, subject to k constraints $g_1(x_1, \ldots, x_n) = 0$, ..., $g_k(x_1, \ldots, x_n) = 0$.

(i) Introduce k Lagrange multipliers $\lambda_1, \ldots, \lambda_k$

(ii) Define the Lagrangian Λ by

$$\Lambda(x, \lambda) = f(x_1, \ldots, x_n) - \sum_{r=1}^{k} \lambda_r \, g_r(x_1, \ldots, x_n)$$

$$= f(x_1, \ldots, x_n) - \lambda_1 g_1(x_1, \ldots, x_n) - \cdots - \lambda_k g_k(x_1, \ldots, x_n).$$

(iii) The stationary points of f subject to the constraints $g_1 = 0, \ldots, g_k = 0$ are precisely the set of values of $(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_k)$ at which

$$\frac{\partial \Lambda}{\partial x_i} = 0, \; i = 1, \ldots, n \text{ and } \frac{\partial \Lambda}{\partial \lambda_r} = 0, \; r = 1, \ldots, k \, .$$

Example Find the maximum value of $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

Solution: We wish to maximize $f(x, y, z) = x + 2y + 3z$ subject to the constraints $g(x, y, z) = x - y + z - 1$ and $h(x, y, z) = x^2 + y^2 - 1$. First we have $\nabla f = \langle 1,2,3 \rangle$, $\nabla h = \langle 2x, 2y, 0 \rangle$. Thus we need to solve the system of equations (3+2):

$$\nabla f = \lambda \nabla g + \mu \nabla h, \; x - y + z = 1, \; x^2 + y^2 = 1$$. That is

$$1 = \lambda + 2x\mu \quad \text{(i)}$$

$$2 = -\lambda + 2y\mu \quad \text{(ii)}$$

$$3 = \lambda + 0 \quad \text{(iii)}$$

$$x - y + z = 1 \quad \text{(iv)}$$

$$x^2 + y^2 \quad = 1 \qquad\qquad\qquad\qquad \text{(v)}$$

From (iii), $\lambda = 3$. Substituting this into (i) and (ii), we get $x = -\frac{1}{\mu}$ and $y = \frac{5}{2\mu}$. Note that $\mu \neq 0$ by (ii) and (iii). From (iv), we have

$$z = 1 - x + y = 1 + \frac{1}{\mu} + \frac{5}{2\mu} = 1 + \frac{7}{2\mu}. \quad \text{(vi)}$$

Using (v), we have $\left(-\frac{1}{\mu}\right)^2 + \left(\frac{5}{2\mu}\right)^2 = 1$. From this, we can solve for μ, giving $\mu = \pm\frac{\sqrt{29}}{2}$. Thus, $x = -\frac{2\sqrt{29}}{29}$ or $x = \frac{2\sqrt{29}}{29}$. The corresponding values of y are $\frac{5\sqrt{29}}{29}, -\frac{5\sqrt{29}}{29}$. Using (vi), the corresponding values of z are $1 + \frac{7\sqrt{29}}{29}, 1 - \frac{7\sqrt{29}}{29}$. Therefore, the two possible extreme values are at points $P_1 = \left(-\frac{2\sqrt{29}}{29}, \frac{5\sqrt{29}}{29}, 1 + \frac{7\sqrt{29}}{29}\right)$ and $P_2 = \left(\frac{2\sqrt{29}}{29}, -\frac{5\sqrt{29}}{29}, 1 - \frac{7\sqrt{29}}{29}\right)$. As $f(P_1) = 3 + \sqrt{29}$ and $f(P_2) = 3 - \sqrt{29}$, the maximum value is $3 + \sqrt{29}$ and the minimum value is $3 - \sqrt{29}$.

We shall now examine more facts about functions of one variable.

2.6 Mathematical Applications

Definition 2.6.1 A function f with domain D is said to have an absolute maximum at c if $f(x) \leq f(c) \quad \forall\ x \in D$. The number $f(c)$ is called the absolute maximum of f on D. The function f is said have a local maximum (or relative maximum) at c if there is some open interval (a, b) containing c and $f(c)$ is the absolute maximum of f on (a, b).

Definition 2.6.2 A function f with domain D is said to have an absolute minimum at c if $f(c) \leq f(x) \quad \forall\ x \in D$. The number $f(c)$ is called the absolute minimum of f on D. The number f is called a local minimum (or relative minimum) of f if there is some open interval (a, b) containing c and $f(c)$ is the absolute minimum of f on (a, b).

Definition 2.6.3 An absolute maximum or absolute minimum of f is called an absolute extremum of f. A local maximum or minimum of f is called a local extremum of f.

Theorem 2.6.4 (Extreme Value Theorem) If a function f is continuous on a closed and bounded interval $[a, b]$, then there exist two points, c_1 and c_2, in $[a, b]$ such that $f(c_1)$ is the absolute minimum of f on $[a, b]$ and $f(c_2)$ is the absolute maximum of f on $[a, b]$.

Definition 2.6.5 A function f is said to be increasing on an open interval (a, b) if $f(x_1) < f(x_2)\ \ \forall\ x_1, x_2 \in (a, b)$ such that $x_1 < x_2$. The function f is said to be decreasing on (a, b) if $f(x_1) > f(x_2)\ \ \forall\ x_1, x_2 \in (a, b)$ such that $x_1 < x_2$. The function f is said to be non – decreasing on (a, b) if $f(x_1) \leq f(x_2)\ \ \forall\ x_1, x_2 \in (a, b)$ such that $x_1 < x_2$. The function f is said to be non – increasing on (a, b) if $f(x_1) \geq f(x_2)\ \ \forall\ x_1, x_2 \in (a, b)$ such that $x_1 < x_2$.

Theorem 2.6.6 Suppose that a function f is defined on some open interval (a, b) containing a number c such that $f'(c)$ exists and $f'(c) \neq 0$. Then $f(c)$ is not a local extremum of f.

Proof: Suppose that $f'(c) \neq 0$. Let $\epsilon = \frac{1}{2}|f'(c)|$. Then $\epsilon > 0$. Since $\epsilon > 0$ and

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

there exists some $\delta > 0$ such that if $0 < |x - c| < \delta$, then

$$\left|\frac{f(x) - f(c)}{x - c} - f'(c)\right| < \frac{1}{2}|f'(c)|$$

$$-\frac{1}{2}|f'(c)| < \frac{f(x) - f(c)}{x - c} - f'(c) < \frac{1}{2}|f'(c)|$$

$$f'(c) - \frac{1}{2}|f'(c)| < \frac{f(x) - f(c)}{x - c} < f'(c) + \frac{1}{2}|f'(c)|.$$

The following three numbers have the same sign, namely, $f'(c),\ f'(c) - \frac{1}{2}|f'(c)|$, and $f'(c) + \frac{1}{2}|f'(c)|$. Since $f'(c) > 0$ or $f'(c) < 0$, we conclude that

$$0 < \frac{f(x) - f(c)}{x - c} \quad \text{or} \quad \frac{f(x) - f(c)}{x - c} < 0 \qquad \forall\ x \text{ such that } 0 < |x - c| < \delta.$$

Thus, if $c - \delta < x_1 < c < x_2 < c + \delta$, then either $f(x_1) < f(c) < f(x_2)$ or $f(x_1) > f(c) > f(x_2)$. It follows that $f(c)$ is not a local extremum. ■

Theorem 2.6.7 If f is defined on an open interval (a, b) containing c, $f(c)$ is a local extremum of f and $f'(c)$ exists, then $f'(c) = 0$.

Theorem 2.6.8 (**Rolle's Theorem**) Suppose that a function f is continuous on a closed and bounded interval $[a, b]$, differentiable on the open interval (a, b) and $f(a) = f(b)$. Then there exists some c such that $a < c < b$ and $f'(c) = 0$.

Proof: Since f is continuous on $[a,b]$, there exist two numbers c_1 and c_2 on $[a,b]$ such that $f(c_1) \leq f(x) \leq f(c_2)$ $\forall$ $x \in [a,b]$. (Extreme Value Theorem) If $f(c_1) = f(c_2)$, then the function f has a constant value on $[a,b]$ and $f'(c) = 0$ for $c = \frac{1}{2}(a+b)$. If $f(c_1) \neq f(c_2)$, then either $f(c_1) \neq f(a)$ or $f(c_2) \neq f(a)$. But $f'(c_1) = 0$ and $f'(c_2) = 0$. It follows that $f'(c_1) = 0$ or $f'(c_2) = 0$ and either c_1 or c_2 is between a and b. ■

Theorem 2.6.9 (**The Mean Value Theorem**) Suppose that a function f is continuous on a closed and bounded interval $[a,b]$ and f is differentiable on the open interval (a,b). Then there exists some number c such that $a < c < b$ and

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$

Proof: We define a function $g(x)$ that is obtained by subtracting the line joining $(a, f(a))$ and $(b, f(b))$ from the function :

$$g(x) = f(x) - \left[\frac{f(b)-f(a)}{b-a}(x-a) + f(a)\right].$$

The g is continuous on $[a,b]$ and differentiable on (a,b). Furthermore, $g(a) = g(b) = 0$. By Rolle's Theorem, there exists some number c such that $a < c < b$ and

$$\begin{aligned} 0 &= g'(c) \\ &= f'(c) - \frac{f(b)-f(a)}{b-a}. \end{aligned}$$

Hence, $\frac{f(b)-f(a)}{b-a} = f'(c)$ as required . ■

Theorem 2.6.10 (**Cauchy – Mean Value Theorem**) Suppose that two functions f and g are continuous on a closed and bounded interval $[a,b]$, differentiable on the open interval (a,b) and $g'(x) \neq 0$ for all $x \in (a,b)$. Then there exists some number c in (a,b) such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof: We define a new function h on $[a,b]$ as follows:

$$h(x) = f(x) - f(a) - \frac{f(b)-f(a)}{g(b)-g(a)}(g(x) - g(a)).$$

Then h is continuous on $[a,b]$ and differentiable on (a,b). Furthermore, $h(a) = 0$ and $h(b) = 0$. By Rolle's Theorem, there exists some c in (a,b) such that $h'(c) = 0$. Then

$$0 = h'(c) = f'(c) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(c)$$

and, hence, $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$ as required. ■

Theorem 2.6.11 (**L'Hospital Rule, $\frac{0}{0}$ Form**) Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval (a,b) containing c (except possibly at c). Suppose that $\lim_{x\to c} f(x) = 0$, $\lim_{x\to c} g(x) = 0$ and $\lim\limits_{x\to c} \frac{f'(x)}{g'(x)} = L$, where L is a real number, ∞ or $-\infty$. Then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)} = L\ .$$

Proof: We define $f(c) = 0$ and $g(c) = 0$. Let $x \in (c,b)$. Then f and g are continuous on $[c,x]$, differentiable on (c,x) and $g'(y) \neq 0$ on (c,x). By the Cauchy Mean Value Theorem, there exists some $y \in (c,x)$ such that

$$\frac{f(x)}{g(x)} = \frac{f(x)-f(c)}{g(x)-g(c)} = \frac{f'(y)}{g'(y)}\ .$$

Then

$$\lim_{x\to c^+} \frac{f(x)}{g(x)} = \lim_{y\to c^+} \frac{f'(y)}{g'(y)} = L\ .$$

Similarly, we can prove that

$$\lim_{x\to c^-} \frac{f(x)}{g(x)} = L\ .$$

Therefore,

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)} = L\ .$$

Note: Theorem 2.6.11 is also valid for the case when $\lim_{x\to c} f(x) = \infty \text{ or} -\infty$ and $\lim_{x\to c} g(x) = \infty \text{ or} -\infty$.

Example: Find each of the following limits using L'Hospital rule:

(i) $\lim\limits_{x\to 1} \frac{2x^4-6x^3+x^2+3}{x-1}$ (ii) $\lim\limits_{x\to 0} \frac{\sin 3x}{\sin 5x}$ (iii) $\lim\limits_{x\to 0} \frac{\tan 2x}{\tan 3x}$ (iv) $\lim\limits_{x\to 0} \frac{\sin x}{x}$ (v) $\lim\limits_{x\to 0} \frac{x}{\sin x}$ (vi) $\lim\limits_{x\to 0} \frac{1-\cos x}{x}$
(vii) $\lim\limits_{x\to 0} x \ln x$

Solution:

(i) $\lim\limits_{x\to 1} \frac{2x^4-6x^3+x^2+3}{x-1} = \lim\limits_{x\to 1} \frac{8x^3-18x^2+2x}{1} = 8(1)^3 - 18(1)^2 + 2(1) = -8$

(ii) $\lim\limits_{x\to 0} \frac{\sin 3x}{\sin 5x} = \lim\limits_{x\to 0} \frac{3\cos 3x}{5\cos 5x} = \frac{3}{5}$

(iii) $\lim_{x\to 0} \frac{\tan 2x}{\tan 3x} = \lim_{x\to 0} \frac{2\sec^2 x}{3\sec^2 x} = \frac{2}{3}$

(iv) $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1$

(v) $\lim_{x\to 0} \frac{x}{\sin x} = \lim_{x\to 0} \frac{1}{\cos x} = 1$

(vi) $\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{\sin x}{1} = 0$

(vii) $\lim_{x\to 0} x\ln x = \lim_{x\to 0} \frac{\ln x}{\left(\frac{1}{x}\right)} = \lim_{x\to 0} \frac{\left(\frac{1}{x}\right)}{\left(\frac{-1}{x^2}\right)} = \lim_{x\to 0}(-x) = 0$

Theorem 2.6.13 Suppose that two functions f and g are continuous on a closed and bounded interval $[a, b]$ and are differentiable on the open interval (a, b). Then the following statements are true:

(i) If $f'(x) > 0$ for each $x \in (a, b)$, then f is increasing on (a, b).

(ii) If $f'(x) < 0$ for each $x \in (a, b)$, then f is decreasing on (a, b).

(iii) If $f'(x) \geq 0$ for each $x \in (a, b)$, then f is non – decreasing on (a, b).

(iv) If $f'(x) \leq 0$ for each $x \in (a, b)$, then f is non – increasing on (a, b).

(v) If $f'(x) = 0$ for each $x \in (a, b)$, then f is constant on (a, b).

§ 3.0 TAYLOR SERIES

The Taylor series of $y = f(x)$ about the point $x = x_0$ is define as

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \frac{(x-x_0)^3}{3!}f'''(x_0) + \cdots + \frac{(x-x_0)^n}{n!}f^{(n)}(x_0) + \cdots \tag{3.0.1}$$

We note that Maclaurin's expansion is a special form of Taylor series about $x_0 = 0$.

Theorem 3.1 (**Taylor's Theorem**) Let $f: [a, b] \to \mathbb{R}$ be n times differentiable on $[a, b]$ with $a < c$ and its nth derivative $f^{(n)}$ is also continuous on $[a, b]$ and differentiable on (a, b). Let $x_0 \in [a, b]$. Then, for each $x \in [a, b]$ with $x_0 \neq x$ there exists c between x and x_0 such that

$$f(x) = f(x_0) + \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

The second term on right hand side is called Taylor series and the last term is called Lagrange remainder.

3.2 Examples

(1) Using Taylor series, expand the function $f(x) = 2x^3 + 4x^2 + 3x + 10$ around the point $x_0 = 2$.

Solution: Recall that $f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \frac{(x-x_0)^3}{3!}f'''(x_0) + \cdots + \frac{(x-x_0)^n}{n!}f^{(n)}(x_0) + \cdots$

So, $f(x) = 48 + (x - 2)(51) + \frac{(x-2)^2}{2!}(40) + \frac{(x-2)^3}{3!}(16)$.

(2) Find a Taylor series of $y(x)$ about $x = 0$ if $y(x) = \frac{1}{(1+x)^q}$.

Solution: Direct substitution reveals that the answer is

$$y(x) = 1 - qx + \frac{(-q)(-q-1)}{2!}x^2 + \frac{(-q)(-q-1)(-q-2)}{3!}x^3 + \cdots \quad .$$

It is possible to use Taylor series to find the sums of many different infinite series. The following examples illustrate this idea.

(3) Find the sum of the following series:

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

Solution: Recall the Taylor series for e^x:

$$1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots = e^x$$

The sum of the given series can be obtained by substituting in $x = 1$:

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots = e\ .$$

(4) Find the sums of the following series:

(a) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ (b) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$

Solution:

(a) Recall that $x - \frac{x}{2} + \frac{x}{3} - \frac{x}{4} + \frac{x}{5} - \cdots = \ln(1+x)$. Substituting in $x = 1$ yields

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \ln(2) \quad .$$

(b) Recall that $x - \frac{x}{3} + \frac{x}{5} - \frac{x}{7} + \frac{x}{9} - \cdots = \tan^{-1}(x)$. Substituting in $x = 1$ yields

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \tan^{-1}(1) = \frac{\pi}{4}.$$

This is known as the Gregory – Leibniz formula for π .

Limit Using Power series

When taking a limit as $x \to 0$, you can often simplify things by substitution in a power series that you know. The following examples illustrate the idea.

(5) Evaluate $\lim\limits_{x\to 0} \frac{\sin x}{x^3}$.

Solution: We simply plug in the Taylor series for $\sin x$.

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin x}{x^3} &= \lim_{x\to 0} \frac{\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{5!}x^7 + \cdots\right) - x}{x^3} \\
&= \lim_{x\to 0} \frac{-\frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{5!}x^7 + \cdots}{x^3} \\
&= \lim_{x\to 0} \left(-\frac{1}{3!} + \frac{1}{5!}x^5 - \frac{1}{7!}x^4 + \cdots\right) \\
&= -\frac{1}{3!} \\
&= -\frac{1}{6} \quad \blacksquare
\end{aligned}$$

(6) Evaluate $\lim\limits_{x\to 0} \frac{x^2 e^x}{\cos x - 1}$

Solution: We simply plug in the Taylor series for e^x and $\cos x$:

$$\lim_{x\to 0} \frac{x^2 e^x}{\cos x - 1} = \lim_{x\to 0} \frac{x^2\left(1 + x + \frac{1}{2}x^2 + \cdots\right)}{\left(1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right) - 1}$$

$$= \lim_{x \to 0} \frac{x^2 + x^3 + \frac{1}{2}x^4 + \cdots}{-\frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots}$$

$$= \lim_{x \to 0} \frac{1 + x + \frac{1}{2}x^2 + \cdots}{-\frac{1}{2} + \frac{1}{4!}x^2 - \frac{1}{6!}x^4 + \cdots}$$

$$= \frac{1}{-\frac{1}{2}}$$

$$= -2 \qquad \blacksquare$$

(7) Evaluate $\lim_{x \to 0} \frac{\ln(\cos x)}{x^2}$.

Solution: Using the Taylor series formula, the first few terms of the Taylor series for $\ln(\cos x)$ are:

$$\ln(\cos x) = -\frac{1}{2}x^2 - \frac{1}{12}x^4 + \cdots$$

Therefore,

$$\lim_{x \to 0} \frac{\ln(\cos x)}{x^2} = \lim_{x \to 0} \frac{-\frac{1}{2}x^2 - \frac{1}{12}x^4 + \cdots}{x^2}$$

$$= \lim_{x \to 0} \left(-\frac{1}{2} - \frac{1}{12}x^2 + \cdots\right)$$

$$= -\frac{1}{2} \qquad \blacksquare$$

Limit as $x \to a$ can be obtained using a Taylor series centred at $x = a$.

(8) Evaluate $\lim_{x \to 0} \frac{\ln x}{x-1}$

Solution: Recall that $\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots$

Plugging this gives

$$\lim_{x \to 0} \frac{\ln x}{x-1} = \lim_{x \to 0} \frac{(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots}{x-1}$$

$$= \lim_{x \to 0} \left(1 - \frac{1}{2}(x-1) + \frac{1}{3}(x-1)^2 - \cdots\right)$$

$$= 1 \qquad \blacksquare$$

3.3. Taylor Polynomials

A partial sum of a Taylor series is called a Taylor polynomial. For illustration, the Taylor polynomials for e^x are :

$$T_0(x) = 1$$

$$T_1(x) = 1 + x$$

$$T_2(x) = 1 + x + \frac{1}{2}x^2$$
$$\vdots$$

You can approximate any function $f(x)$ by its Taylor polynomial: $f(x) \approx T_n(x)$. If you use the Taylor polynomial centred at $x = a$.

Definition 3.3.1: (Taylor Polynomial) Let $f(x)$ be a function. The Taylor polynomials for $f(x)$ centred at $x = a$ are:

$$T_0(x) = f(a)$$
$$T_1(x) = f(a) + f'(a)(x - a)$$
$$T_2(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2$$
$$\vdots$$

Note: The 1st – degree Taylor polynomial is just the tangent line to $f(x)$ at $x = a$: $T_1(x)$. This is often called the linear approximation to $f(x)$ near $x = a$. 2nd – degree = quadratic approximation.

Example 9:

(a) Find the 5th – degree Taylor polynomial for $\sin x$.

(b) Use the answer in (a) to approximate $\sin(0.2)$.

Solution:

(a) This is just to find all terms of the Taylor series up to x^5:

$$T_5(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$$
$$= x - \frac{1}{6}x^3 + \frac{1}{120}x^5$$

(b) $\sin(0.2) \approx T_5(0.2)$

$$= (0.2) - \frac{1}{6}(0.2)^3 + \frac{1}{120}(0.2)^5$$
$$= 0.198669$$

Exercise 3

1. Evaluate the following limits:

(i) $\lim_{x\to 0}\frac{\cos x - 1}{x^2}$ (ii) $\lim_{x\to 0}\frac{x}{e^{3x}-1}$ (iii) $\lim_{x\to 0}\frac{\sin 4x}{x}$ (iv) $\lim_{x\to 0}\frac{\ln(1+x^2)}{x^2}$ (v) $\lim_{x\to 0}\frac{\ln x}{\sqrt{x-1}}$

2. Find the sum of the given series.

$$1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \cdots$$

3. (a) Find the 3rd – degree Taylor polynomial for the function $f(x) = \ln x$ centred at $a = 1$

   (b) Use your answer from part (a) to approximate $\ln(1.15)$.

4. (a) Find the quadratic approximation for the function $f(x) = x^{2/3}$ centred at $x_0 = 4$.

   (b) Use your answer from part (a) to approximate $(4.2)^{2/3}$.

5. Find the 4th – degree Taylor polynomial for e^{-x}.

§ 4.0 INTEGRATION (ANTIDIFFERENTIATION)

The process of finding a function $g(x)$ such that $g(x) = f(x)$, for a given $f(x)$, is called antidifferentiation.

Definition 4.1.1: Let f and g be two continuous functions defined on an open interval (a, b). If $g'(x) = f(x)$ for each $x \in (a, b)$, then g is called an antiderivative (integral) of f on (a, b).

Theorem 4.1.2: If $g_1(x)$ and $g_2(x)$ are any two antiderivatives of $f(x)$ on (a, b), then there exists some constant C such that

$$g_1(x) = g_2(x) + C \quad .$$

Proof: If $h(x) = g_1(x) - g_2(x)$, then

$$\begin{aligned} h'(x) &= g'_1(x) - g'_2(x) \\ &= f(x) - f(x) \\ &= 0 \quad \forall \quad x \in (a, b) \end{aligned}$$

By Theorem 2.6.13, part (iv), there exists some constant c such that for all x in (a, b),

$$C = h(x) = g_1(x) - g_2(x)$$

$$g_2(x) = g_1(x) + C.$$

Definition 4.1.3: If $g(x)$ is an antiderivative of f on (a, b), then the set $\{g(x) + C : C \text{ is a constant}\}$ is called a one – parameter family of antiderivatives of f. We called this one – parameter family of antiderivatives the indefinite integral of $f(x)$ on (a, b) and write $\int f(x)dx = g(x) + C.$ $\qquad (C \equiv \int 0dx)$

Note that: $\frac{d}{dx}(\int f(x)dx) = g'^{(x)} = f(x).$

4.1.4 Example: The following statements are true:

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C\,,\ n \neq -1$
2. $\int x^6\, dx = \frac{1}{7}x^7 + C$
3. $\int \frac{1}{x} dx = \ln x + C$
4. $\int \sin x\, dx = -\cos x + C$

5. $\int \sin(ax)\, dx = \frac{-1}{a}\cos(ax) + C$
6. $\int \cos x\, dx = \sin x + C$
7. $\int \cos(ax)\, dx = \frac{1}{a}\sin(ax) + C$
8. $\int e^x\, dx = e^x + C$
9. $\int e^{ax}\, dx = \frac{1}{a}e^{ax} + C$
10. $\int \tan x\, dx = \ln|\sec x| + C$
11. $\int \sec x\, dx = \ln|\sec x + \tan x| + C$
12. $\int \csc x\, dx = -\ln|\csc x + \cot x| + C$
13. $\int \sec^2(ax)\, dx = \frac{1}{a}\tan(ax) + C$

4.2 The Definite Integral

Definition 4.2.1: If f is continuous on $[a, b]$ and $L_f = U_f = I$, then we say that:

(i) f is Integrable on $[a, b]$;

(ii) the definite integral of $f(x)$ form $x = a$ to $x = b$ is I;

(iii) I is expressed in symbol, by the equation $I = \int_a^b f(x)\, dx$;

(iv) If $f(x) \geq 0$ for each $x \in [a, b]$, then the area, A, bounded by the curves $y = f(x)$, $y = 0, x = a$ and $x = b$, is defined to be the definite integral of $f(x)$ from $x = a$ to $x = b$. That is, $A = \int_a^b f(x)\, dx$.

(v) For convenience, we define $\int_a^a f(x)\, dx = 0,\quad \int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$.

Theorem 4.2.2: (Linearity) Suppose that f and g are continuous on $[a, b]$ and c_1 , c_2 constants. Then

(i) $\int_a^b (f(x) + g(x))\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$

(ii) $\int_a^b (f(x) - g(x))\, dx = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx$

(iii) $\int_a^b c_1 f(x)\, dx = c_1 \int_a^b f(x)\, dx,\ \int_a^b c_2 g(x)\, dx = c_2 \int_a^b g(x)\, dx$ and
$\int_a^b (c_1 f(x) + c_2 g(x))\, dx = c_1 \int_a^b f(x)\, dx + c_2 \int_a^b g(x)\, dx$

Theorem 4.2.3: (Additive) If f is continuous on $[a, b]$ and $a < c < b$, then

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

Theorem 4.2.4: (Order Property) If f and g are continuous on $[a,b]$ and $f(x) \leq g(x)$ for all $x \in [a.b]$, then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

Proof: Suppose that f and g are continuous on $[a,b]$ and $f(x) \leq g(x)\ \forall\ \ x \in [a,b]$. For each i there exist numbers $c_i, c^*{}_i, d_i,$ and $d^*{}_i$ such that

$f(c_i)$ = absolute minimum of f on $[x_{i-1}, x_i]$,

$f(d_i)$ = absolute maximum of f on $[x_{i-1}, x_i]$,

$g(c^*{}_i)$ = absolute minimum of g on $[x_{i-1}, x_i]$,

$g(d^*{}_i)$ = absolute maximum of g on $[x_{i-1}, x_i]$.

By the assumption that $f(x) \leq g(x)$ on $[a,b]$, we get

$f(c_i) \leq g(c^*{}_i)$ and $f(d_i) \leq g(d^*{}_i)$

Hence,

$L_f \leq L_g$ and $U_f \leq U_g$.

It follows that

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx\,. \qquad \blacksquare$$

Theorem 4.2.5 (Mean Value Theorem for Integrals) If f is a continuous on $[a,b]$, then there exists some point c in $[a,b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a).$$

Proof: Suppose that f is continuous on $[a,b]$, and $a < b$. Let m = absolute minimum of f on $[a,b]$, and M = absolute maximum of f on $[a,b]$. Then by Theorem 4.2.4,

$$m \leq \frac{1}{b-a}\int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx = M(b-a)$$

And $\qquad m \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq M.$

By the Intermediate value theorem for continuous functions, there exists some c such that $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$ and $\int_a^b f(x)\,dx = f(c)(b-a)$. For $a = b,$ take $c = a$. $\blacksquare$

Theorem 4.2.6 (Fundamental Theorem of Calculus, 1st Form) Suppose that f is continuous on some closed and bounded interval $[a,b]$ and $g(x) = \int_a^x f(t)\,dt$ for each $x \in [a,b]$. Then $g(x)$ is continuous on $[a,b]$, differentiable on (a,b) and for all $x \in (a,b)$, $g'(x) = f(x)$. That is, $\quad \frac{d}{dx}\left[\int_a^x f(t)\right]dt = f(x).$

Theorem 4.2.7 (Fundamental Theorem of Calculus, 2nd Form) If f and g are continuous on a closed and bounded interval $[a, b]$ and $g'(x) = f(x)$ on $[a, b]$, then

$$\int_a^b f(x)\,dx = g(b) - g(a).$$

We use the notation: $[g(x)]_a^b = g(b) - g(a)$.

4.2.8 Examples

Compute each of the following definition integrals

1. (i) $\int_0^4 x^2\,dx$ (ii) $\int_0^\pi \sin x\,dx$ (iii) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\,dx$ (iv) $\int_0^{10} e^x\,dx$ (v) $\int_0^{\frac{\pi}{3}} \tan x\,dx$ (vi) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot x\,dx$

   (vii) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec x\,dx$ (viii) $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \csc x\,dx$ (ix) $\int_0^1 \sinh x\,dx$ (x) $\int_0^1 \cosh x\,dx$ (xi) $\int_1^{20} \frac{1}{x}\,dx$

Solution:

(i) $\int_0^4 x^2\,dx = \left[\frac{x^3}{3}\right]_0^4 = \left[\frac{4^3}{3}\right] - \left[\frac{0^3}{3}\right] = \frac{64}{3}$

(ii) $\int_0^\pi \sin x\,dx = [-\cos x]_0^\pi = 1 - (-1) = 2$

(iii) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\,dx = [\sin x]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 1 - (-1) = 2$

(iv) $\int_0^{10} e^x\,dx = [e^x]_0^{10} = e^{10} - 1$

(v) $\int_0^{\frac{\pi}{3}} \tan x\,dx = [\ln|\sec x|]_0^{\frac{\pi}{3}} = \ln\left|\sec\left(\frac{\pi}{3}\right)\right| = \ln 2$

(vi) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot x\,dx = [\ln|\sin x|]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \ln(1) - \ln\left(\frac{1}{2}\right)$

(vii) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec x\,dx = [\ln|\sec x + \tan x|]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \ln|\sqrt{2} + 1| - \ln|\sqrt{2} - 1|$

(viii) $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \csc x\,dx = [-\ln|\csc x + \cot x|]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} = -\ln|\sqrt{2} - 1| + \ln|\sqrt{2} + 1|$

(ix) $\int_0^1 \sinh x\,dx = [\cosh x]_0^1 = \cosh 1 - 1$

(x) $\int_0^1 \cosh x\,dx = [\sinh x]_0^1 = \sinh 1$

(xi) $\int_1^{20} \frac{1}{x}\,dx = [\ln|x|]_1^{20} = \ln(20)$

2. Verify each of the following: (i) $\int_0^4 x^2\,dx = \int_0^3 x^2\,dx + \int_3^4 x^2\,dx$

   (ii) $\int_1^4 x^2\,dx < \int_1^4 x^3\,dx$

Solution:

(i) $\int_0^4 x^2\,dx = \left[\frac{x^3}{3}\right]_0^4 = \frac{64}{3}$

$$\int_0^3 x^2\,dx + \int_3^4 x^2\,dx = \left[\frac{x^3}{3}\right]_0^3 + \left[\frac{x^3}{3}\right]_3^4$$
$$= \left(\frac{27}{3} - 0\right) + \left(\frac{64}{3} - \frac{27}{3}\right)$$
$$= \frac{64}{3}.$$

Therefore,

$$\int_0^4 x^2\,dx = \int_0^3 x^2\,dx + \int_3^4 x^2\,dx.$$

(ii) $\int_1^4 x^2\,dx = \left[\frac{x^3}{3}\right]_1^4 = \frac{64}{3} - \frac{1}{3} = 21$

$$\int_1^4 x^3\,dx = \left[\frac{x^3}{4}\right]_1^4 = \left(64 - \frac{1}{4}\right) = \frac{255}{4}$$

Therefore,

$\int_1^4 x^2\,dx < \int_1^4 x^3\,dx$. We observe that $x^2 < x^3$ on $[1,4]$.

4.2.8 Integration by Substitution

Many functions are formed by using compositions. In dealing with a composite function, it is useful to change variables of integration. It is convenient to use the following differential notation: If $u = g(x)$, then $du = g'(x)dx$.

Example:

(1) Evaluate the following integrals: (i) $\int 2xe^{x^2}dx$ (ii) $\int_0^2 \sin(3x)\,dx$ (iii) $\int_0^2 3x\cos(x^2)\,dx$

(2) Determine the area, A, bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$ and $x = \pi$.

Solution:

1. (i) Let $u = x^2$. Then $\frac{du}{dx} = 2x \quad \Rightarrow dx = \frac{du}{2x}$. So, we have

$$\int 2xe^{x^2}dx = \int 2x\,e^u\left(\frac{du}{2x}\right)$$
$$= \int e^u\,du$$
$$= e^u + C$$
$$= e^{x^2} + C.$$

(ii) Set $u = 3x$. At $= 0, u = 0$ and at $x = 2, u = 6$. Then $\frac{du}{dx} = 3 \quad \Rightarrow dx = \frac{du}{3}$. So, we have

$$\int_0^2 \sin(3x)\,dx = \int_0^6 \sin u\left(\frac{du}{3}\right)$$
$$= \frac{1}{3}\int_0^6 \sin u\,du$$
$$= \frac{1}{3}[-\cos u]_0^6$$

$$= \frac{1}{3}(1 - \cos 6)$$

(iii) $\int_0^2 3x\cos(x^2)\,dx = \int_0^4 \cos u \left(\frac{3}{2}du\right)$

$$= \frac{3}{2}[\sin u]_0^4$$

$$= \frac{3}{2}\sin 4,$$

where $u = x^2,\ du = 2xdx$.

2. We note that $\cos x \geq \sin x$ on $\left[0, \frac{\pi}{4}\right]$ and $\sin x \geq \cos x$ on $\left[\frac{\pi}{4}, \pi\right]$. Therefore, the area is given by

$$A = \int_0^{\pi} |\sin x - \cos x|\,dx$$

$$= \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\,dx + \int_{\frac{\pi}{4}}^{\pi} (\sin x - \cos x)\,dx$$

$$= [\sin x + \cos x]_0^{\frac{\pi}{4}} + [-\cos x - \sin x]_{\frac{\pi}{4}}^{\pi}$$

$$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 1\right) + \left(1 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right)$$

$= 2\sqrt{2}$ square units.

<u>4.2.9 Integration by Parts</u>

The product rule of differentiation yields an integration technique known as integration by parts. Let us begin with the product rule:

$$\frac{d}{dx}(u(x)v(x)) = \frac{du(x)}{dx}v(x) + u(x)\frac{dv(x)}{dx}.$$

On integrating each term with respect to x from $x = a$ to $x = b$, we get

$$\int_a^b \frac{d}{dx}(u(x)v(x))\,dx = \int_a^b v(x)\left(\frac{du(x)}{dx}\right)dx + \int_a^b u(x)\left(\frac{dv(x)}{dx}\right)dx.$$

By using the differential notation and the fundamental theorem of calculus, we have

$$[u(x)v(x)]_a^b = \int_a^b v(x)u'(x)\,dx + \int_a^b u(x)\,v'(x)dx.$$

The standard form of this integration by parts formula is written as:

(i) $\int_a^b u(x)v'(x)\,dx = [u(x)v(x)]_a^b - \int_a^b v(x)\,u'(x)dx$ and

(ii) $\int u\,dv = uv - \int v\,du$

<u>Example</u>: Evaluate the following integrals:

(i) $\int x\sin x\,dx$ (ii) $\int x^2 e^x dx$ (iii) $\int_0^{\frac{\pi}{2}} x\cos x\,dx$ (iv) $\int x\,e^{-x}dx$ (v) $\int(\ln x)\,dx$

<u>Solution</u>: (ii) and (v) left as exercise.

(i) We set $u = x$ and $dv = \sin x\, dx$. Then $du = dx$ and $v(x) = \int \sin x\, dx = -\cos x$ (we drop constant C since we are yet to finish the required integral). Then, by the integration by parts, we have

$$\begin{aligned}\int x \sin x\, dx &= \int u\, dv \\ &= uv - \int v\, du \\ &= x(-\cos x) - \int (-\cos x)\, dx \\ &= -x\cos x + \sin x + C.\end{aligned}$$

(iii) $$\begin{aligned}\int_0^{\frac{\pi}{2}} x \cos x\, dx &= [x \sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x\, dx \\ &= \left[\frac{\pi}{2}(1) - 0(0)\right] - [-\cos x]_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{2} - 1\ .\end{aligned}$$

(iv) $$\begin{aligned}\int x\, e^{-x} dx &= x(e^{-x}) - \int (-e^{-x})\, dx \\ &= -xe^{-x} - e^{-x} + C.\end{aligned}$$

4.2.10 Volume, Arc length and Surface Area

Let f be a function that is continuous on $[a, b]$. Let R denote the region bounded by the curves $x = a$, $x = b$, $y = 0$ and $y = f(x)$. Then, the **volume** V obtained by rotating R about the $x-$axis is given by

$$V = \int_a^b \pi\, (f(x))^2 dx \text{ or } \int_a^b \pi\, y^2 dx.$$

If we rotate the plane region described by $f(x) \le y \le g(x)$ and $a \le x \le b$ around the y-axis, the volume of the resulting solid is

$$V = \int_a^b 2\pi\, x(g(x) - f(x))dx\ ,$$

or using $V = \int_a^b \pi\, x^2 dy$ when both the lower and upper limit along y-axis are known.

Example

1. Find the volume of a sphere of radius r.

Solution: We recall that the equation of a circle about origin is $x^2 + y^2 = r^2$, therefore we have $$\begin{aligned}V &= \int_{-r}^{r} \pi\, (r^2 - x^2)dx \\ &= \frac{4}{3}\pi r^2 \text{ cubic units.}\end{aligned}$$

2. A solid is formed by the rotation about OY of the part of the curve $y = x^3$ between $y = 1$

and $y = 8$. Show that the volume is $\frac{93\pi}{5}$ cubic units.

Proof: $V = \int_1^8 \pi x^2 dy$

$= \pi \int_1^8 \left(y^{\frac{1}{3}}\right)^2 dy$

$= \frac{3\pi}{5}\left[y^{\frac{5}{3}}\right]_1^8$

$= \frac{3}{5}\pi(8)^{\frac{5}{3}} - \frac{3}{5}\pi(1)^{\frac{5}{3}} = \frac{93\pi}{5}$ cubic units.

3. Consider the region R bounded by $y = \sin x$, $y = 0, x = 0$ and $x = \pi$. Find the volume generated when R rotated about (a) x-axis (b) y-axis (c) $x = \pi$.

   Solution:

   (i) $V = \int_0^\pi \pi \sin^2 x \, dx$

   $= \pi\left[\frac{1}{2}(x - \sin x \cos x)\right]_0^\pi$

   $= \frac{\pi^2}{2}$.

   (ii) $V = \int_0^\pi 2\pi x \sin x \, dx$

   $= 2\pi[-x\cos x + \sin x]_0^\pi$ (using integration by part)

   $= 2\pi(\pi)$

   $= 2\pi^2$.

   (iii) $V = \int_0^\pi 2\pi(\pi - x)\sin x \, dx$

   $= 2\pi[-\pi\cos x + x\cos x - \sin x]_0^\pi$

   $= 2\pi[2\pi - \pi]$

   $= 2\pi^2$.

The **arc length,** L is calculated using the formula:

$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

Example

1. Let $C = \{(x, \cosh x): 0 \le x \le 2\}$. Then the arc length L of C is given by

$$L = \int_0^2 \sqrt{1 + \sinh^2 x} \, dx$$

$$= \int_0^2 \cosh x \, dx$$
$$= [\sinh x]_0^2$$
$$= \sinh 2.$$

2. Let $C = \left\{\left(x,\ \tfrac{2}{3}x^{3/2}\right): 0 \le x \le 4\right\}$. Then the arc length L of the curve C is given by

$$L = \int_0^4 \sqrt{1 + \left(\tfrac{2}{3} \cdot \tfrac{3}{2} x^{1/2}\right)^2} dx$$
$$= \int_0^4 (1+x)^{1/2} dx$$
$$= \left[\tfrac{2}{3}(1+x)^{3/2}\right]_0^4$$
$$= \frac{2}{3}\left[5\sqrt{5} - 1\right]$$

3. Prove that the circumference of a circle of radius r is $2\pi r$.

<u>Proof</u>: The equation of the circle at the origin is $x^2 + y^2 = r^2$. Differentiating with respect to x, we have

$$2x + 2y\frac{dy}{dx} = 0 \quad \Rightarrow \frac{dy}{dx} = \frac{x}{y}.$$

So,

$$L = \int_0^r \sqrt{1 + \left(\frac{x}{y}\right)^2} \, dx$$
$$= \int_0^r \sqrt{\frac{x^2+y^2}{y^2}} dx$$
$$= \int_0^r \frac{r}{y} dx$$
$$= \int_0^r \frac{r}{\sqrt{r^2-x^2}} dx$$
$$= \left[r \sin^{-1}\frac{x}{r}\right]_0^r$$
$$= r \sin^{-1} 1$$
$$= r \cdot \frac{\pi}{2}$$

Hence, the circumference of the circle is $4 \cdot \frac{\pi r}{2} = 2\pi r$. ■

The **<u>surface area</u>** S_x generated by rotating C about the x-axis is given by

$$S_x = \int_a^b 2\pi \, |f(x)| \sqrt{1 + (f'(x))^2} dx \, .$$

While the surface area S_y generated by rotating C about the y-axis is given by

$$S_y = \int_a^b 2\pi \, |x| \sqrt{1 + (f'(x))^2} dx \, .$$

Example: Let $C = \{(x,\ \cosh x) : 0 \le x \le 4\}$.

1. The surface area S_x generated by rotating C around the x-axis is given by

$$S_x = \int_0^4 2\pi \cosh x \sqrt{1 + \sinh^2 h}\, dx$$
$$= 2\pi \int_0^4 \cosh^2 x\, dx$$
$$= 2\pi \left[\tfrac{1}{2}(x + \sinh x \cosh x)\right]_0^4$$
$$= \pi[4 + \sinh 4 \cosh 4].$$

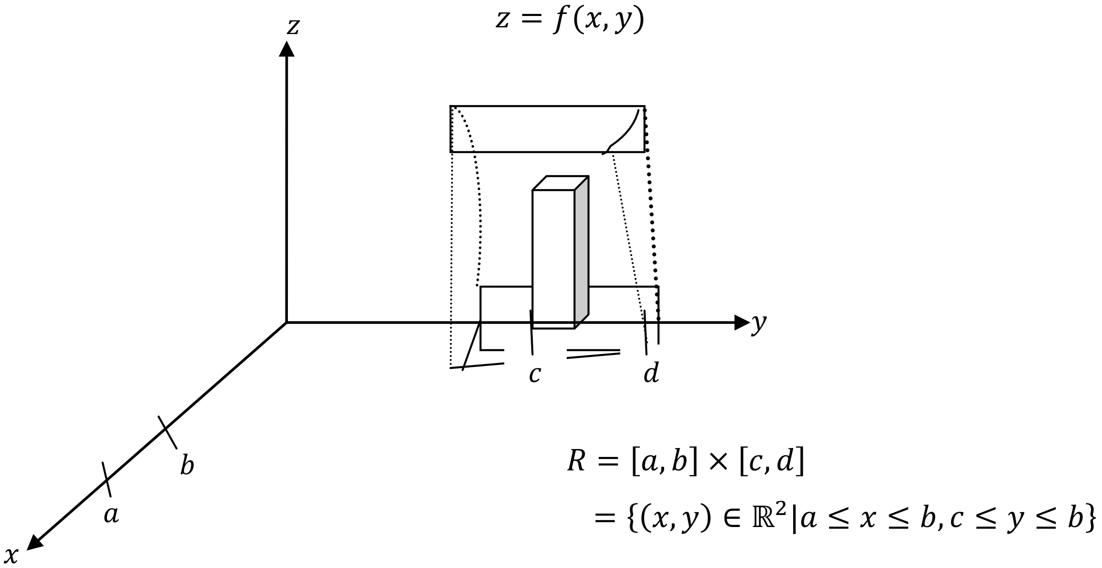
2. The surface area S_y generated by rotating the curve C about the y-axis is given by

$$S_y = \int_0^4 2\pi x \sqrt{1 + \sinh^2 h}\, dx$$
$$= 2\pi \int_0^4 x \cosh^2 x\, dx$$
$$= 2\pi [x \sinh x - \cosh x]_0^4$$
$$= 2\pi[4 \sinh 4 - \cosh 4 + 1]$$

§ 5.0 MULTIPLE INTEGRAL

5.1.1 Volume and Double Integrals

Let f be a function of two variables defined over a rectangle $R = [a, b] \times [c, d]$. We would like to define the double integral of f over R as the (algebraic) volume of the solid under the graph of $z = f(x, y)$ over R.



To do so,, we first subdivide R into mn small rectangles R_{ij} each having area ΔA, where $i = 1,2,3, \dots, m$ and $j = 1,2,3, \dots, n$. For each pair (i, j), we pick an arbitrary point $\left(x^*{}_{ij}, y^*{}_{ij}\right)$ inside R_{ij}. We then use the value $f\left(x^*{}_{ij}, y^*{}_{ij}\right)$ as the height of a rectangular solid erected over R_{ij}. Thus its volume is $\left(x^*{}_{ij}, y^*{}_{ij}\right)\Delta A$. the sum of the volume of all these small rectangular solids approximates the volume of the solid under the graph of $z = f(x, y)$ over R. This sum $\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x^*{}_{ij}, y^*{}_{ij}\right)\Delta A$ is called a Riemann sum of f. We define the **double integral** of f over R as the limit of the Riemann sum as m and n tend to infinity. In other words,

$$\iint_R f(x, y)\, dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x^*{}_{ij}, y^*{}_{ij}\right)\Delta A$$

if this limit exists.

<u>Theorem 5.1.2</u>: If $f(x, y)$ is continuous on R, then $\iint_R f(x, y)\, dA$ always exists.

If $f(x, y) \ge 0$, then the volume V of the solid lies above the rectangle R and below the surface $z = f(x, y)$ is

$$V = \iint_R f(x, y)\, dA .$$

5.2 Iterated Integrals

Let $f(x,y)$ be a function defined on $R = [a,b] \times [c,d]$. We write $\int_c^d f(x,y)\,dy$ to mean that x is regarded as a constant and $f(x,y)$ is integrated with respect to y from $y = c$ to $y = d$. Therefore, $\int_c^d f(x,y)\,dy$ is a function of x and we can integrate it with respect to x from $x = a$ to $x = b$. The resulting integral $\int_a^b \int_c^d f(x,y)\,dy\,dx$ is called an iterated integral. Similarly, one can define the iterated $\int_c^d \int_a^b f(x,y)\,dx\,dy$.

Consider a positive function $f(x,y)$ defined on a rectangle $R = [a,b] \times [c,d]$. Let V be the volume of the solid under the graph of f over R. We may compute V by means of either one of the iterated integrals: $\int_a^b \int_c^d f(x,y)\,dy\,dx$ or $\int_c^d \int_a^b f(x,y)\,dx\,dy$.

Example Evaluate the iterated integrals (a) $\int_0^3 \int_1^2 x^2 y\,dy\,dx$ (b) $\int_1^2 \int_0^3 x^2 y\,dx\,dy$.

Solution:

(a) $\int_0^3 \int_1^2 x^2 y\,dy\,dx = \int_0^3 \left[\frac{x^2y^2}{2}\right]_{y=1}^{y=2} dx$

$= \int_0^3 \frac{3x^2}{2}\,dx$

$= \left[\frac{x^3}{2}\right]_{x=0}^{x=3}$

$= \frac{27}{2}.$

(b) $\int_1^2 \int_0^3 x^2 y\,dx\,dy = \int_1^2 \left[\frac{x^2y}{3}\right]_{x=0}^{x=3} dy$

$= \int_1^2 9y\,dy$

$= \left[\frac{9y^2}{2}\right]_{y=1}^{y=2}$

$= \frac{27}{2}.$

Theorem 5.2.1 (Fubini's Theorem) If $f(x,y)$ is continuous on $R = [a,b] \times [c,d]$, then

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy \quad .$$

5.2.2 Examples

1. Given that $R = \left[0, \frac{\pi}{2}\right] \times \left[0, \frac{\pi}{2}\right]$, evaluate $\iint_R \sin x \cos y\,dA$.

Solution: $\iint_R \sin x \cos y \, dA = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin x \cos y \, dy \, dx$

$$= \int_0^{\frac{\pi}{2}} \sin x \, dx \int_0^{\frac{\pi}{2}} \cos y \, dy$$

$$= [-\cos x]_0^{\frac{\pi}{2}} [\sin y]_0^{\frac{\pi}{2}}$$

$$= [0+1][1-0]$$

$$= 1.$$

Remark: In general, if $f(x,y) = g(x)h(y)$, then

$$\iint_R g(x)\, h(y) dA = \left(\int_a^b g(x)\, dx\right)\left(\int_c^d h(y) dy\right) \text{ where } R = [a,b] \times [c,d]$$

2. Evaluate the double integral $\iint_R (x^2 + y^2)\, dxdy$ where R is the region in xy plane bounded by $y = x^2, x = 2$ and $y = 1$.

Solution: $\int_1^2 \int_1^{x^2} (x^2 + y^2)\, dydx = \int_1^2 \left[x^2 y + \frac{y^3}{3}\right]_{y=1}^{y=x^2} dx$

$$= \int_1^2 \left(x^4 + \frac{x^6}{3} - x^2 - \frac{1}{3}\right) dx$$

$$= \left[\frac{x^5}{5} + \frac{x^7}{21} - \frac{x^3}{3} - \frac{x}{3}\right]_{x=1}^{x=2}$$

$$= \frac{1006}{105}.$$

3. Evaluate (a) $\int_\pi^{2\pi} \int_0^\pi (\sin x + \cos y)\, dxdy$ (b) $\int_1^2 \int_y^{y^2} dxdy$.

Solution:

(a) $\int_0^{2\pi} \int_0^\pi (\sin x + \cos y)\, dxdy = \int_\pi^{2\pi} [(-\cos x) + x \cos y]_{x=0}^{x=\pi} dy$

$$= \int_\pi^{2\pi} [\pi \cos y + 1] dy$$

$$= [\pi \sin y + y]_\pi^{2\pi}$$

$$= 2\pi - \pi - \pi$$

$$= 0.$$

(b)) $\int_1^2 \int_y^{y^2} dxdy = \int_1^2 [x]_{x=y}^{x=y^2} dy$

$$= \int_1^2 (y^2 - y)\, dy$$

$$= \left[\frac{y^3}{3} - \frac{y^2}{2}\right]_1^2$$

$$= \frac{5}{6}.$$

5.2.3 Double Integral over General Region

Let $f(x, y)$ be a continuous function defined on a closed and bounded region D in $\mathbb{R}^2$. The double integral $\iint_D f(x, y)\, dA$ can be defined similarly as the limit of a Riemann sum and iterated integral can also be adopted. In particular, if D is one of the following two types of region in $\mathbb{R}^2$, then we may set up the corresponding iterated integral:

(i) If D is the region bounded by two curves $y = g_1(x)$ and $y = g_2(x)$ from $x = a$ to $x = b$, where $g_2(x) \geq g_1(x)\ \forall\ x \in [a, b]$, we called it a type 1 region and can be computed using iterated integral.

(ii) If D is the region bounded by two curves $x = h_1(y)$ and $x = h_2(y)$ from $y = c$ to $y = d$, where $h_2(y) \geq h_1(y)\ \forall\ y \in [c, d]$, we called it a type 2 region and iterated integral can be computed as well.

Example

1. Evaluate $\iint_D (x + 2y)dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

   Solution: The region D is a type 1. Equating the two parabolas to obtain limits for x, we have $x = \pm 1$. So

$$\iint_D (x + 2y)dA = \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x + 2y)\, dydx$$
$$= \int_{-1}^{1} [xy + y^2]_{y=2x^2}^{y=1+x^2} dx$$
$$= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)\, dx$$
$$= \frac{32}{15}.$$

2. Evaluate the iterated $\iint_D xy\, dA$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

   Solution; Left as exercise.

3. Find the volume of the solid S that is bounded by the curve $x^2 + 2y^2 + z = 16$, the planes $x = 2, y = 2$, and the three coordinate planes.

   Solution: $V = \iint_R (16 - x^2 + 2y^2)\, dA$

$$= \int_0^2 \int_0^2 (16 - x^2 + 2y^2)\, dxdy$$
$$= 48.$$

4. Find the volume of the solid above the xy-plane and is bounded by the cylinder $x^2 + y^2 = 1$ and the plane $z = 0$ and $z = y$.

Solution: Since the plane $z = y$ is the top face of the solid, we may use the function defining this as the height function of this solid. The function whose graph is the plane $z = y$ is simply $f(x, y) = y$. Therefore, the volume of the solid can be computed by integrating this f over the bottom face of the solid which is the semi-circular disk $D = \{(x, y)|x^2 + y^2 \leq 1, y \geq 0\}$.

$$Volume\ (V) = \iint_D f(x,y)\, dA$$
$$= \int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y\, dxdy$$
$$= \int_0^1 2y\sqrt{1-y^2} dy$$
$$= \frac{2}{3}.$$

Properties of Double Integrals

1. $\iint_D (f(x,y) + g(x,y))\, dA = \iint_D f(x,y)\, dA + \iint_D g(x,y)\, dA.$
2. $\iint_D cd(x,y)\, dA = c\iint_D f(x,y)\, dA$, where c is a constant.
3. If $f(x,y) \geq g(x,y)\ \forall\ (x,y) \in D$, then $\iint_D f(x,y)\, dA \geq \iint_D g(x,y)\, dA.$
4. $\iint_D f(x,y)\, dA = \iint_{D_1} f(x,y)\, dA + \iint_{D_2} f(x,y)\, dA,$ where $D = D_1 \cup D_2$ and $D_1 \cap D_2 = \emptyset$ except at their boundary.
5. $\iint_D dA = A(D)$, the area of D.
6. If $m \leq f(x,y) \leq M\ \forall\ (x,y) \in D$, then $mA(D) \leq \iint_D f(x,y)\, dA \leq MA(D)$.

Theorem 5.2.4 (Fubini's Theorem for triple integrals) If $f(x, y, z)$ is continuous on $B = [a, b] \times [c, d] \times [r, s]$, then $\iiint_B f(x,y,z)\, dV = \int_r^s \int_a^b \int_c^d f(x,y,z)\, dydxdz$

Example

1. Evaluate $\iiint_B xyz^2\, dV$, where $B = [0,1] \times [-1,2] \times [0,3]$.

Solution: $\iiint_B xyz^2\, dV = \int_0^3 \int_{-1}^2 \int_0^1 xyz^2\, dxdydz$

$$= \int_0^3 \int_{-1}^2 \left[\frac{x^2yz^2}{2}\right]_{x=0}^{x=1} dydz$$

$= \int_0^3 \int_{-1}^2 \frac{yz^2}{2} dydz$

$= \int_0^3 \frac{3z^2}{4} dz$

$= \frac{27}{4}.$

2. Evaluate $\iiint_B xyz\, dV$ where $B = \{(x, y, z) \in \mathbb{R}^3 |\ 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$.

   Ans. $= \frac{1}{8}$.

Exercise 4

1. Evaluate the following: (a) $\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} y\, dxdy$ (b) $\int_0^{\frac{\pi}{4}} \int_0^1 \sqrt{1 - r^2 \cos^2 \theta}\, r\, drd\theta$.
2. Find the volume of the solid that lies under the curve $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.
3. Evaluate $\iint_R (3x + 4y^2)\, dA$, where R is the region in the upper half plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.
4. Evaluate the following triple integrals:

   (a) $\int_1^4 \int_{-2}^3 \int_0^1 (x^2 + yx + z^3)\, dydxdz$

   (b) $\iiint_B e^{xyz} \cos xyz\, dV$ where $B = [0,2] \times [-1,2] \times [0,3]$.

§ 6.0 LINE INTEGRALS

Consider a plane curve $C: x = x(t),\ y = y(t),\ z = z(t)$ or $r(t) = x(t)i + y(t)j$. We assume C is a smooth curve, meaning that $r'(t) \neq 0$, and $r'(t)$ is continuous for all t. Let $f(x, y)$ be a continuous function defined in a domain containing C.

To define the line integral of f along C, we subdivided arc from $r(a)$ to $r(b)$ into n small arcs of length Δs_i, $i = 1,2, \dots, n$. Pick an arbitrary point $\left(x^*{}_i, y^*{}_j\right)$ inside the ith small arc and form the Riemann sum $\sum_{i=1}^{n} f\left(x^*{}_i, y^*{}_j\right) \Delta s_i$. The line integral of f along C is the limit of this Riemann sum.

Definition 6.1.1 The integral of f along C is define to be

$$\int_C f(x, y)\, ds = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x^*{}_i, y^*{}_j\right) \Delta s_i.$$

We can pull back the integral to an integral in terms of t using the parameterization r. Recall that the arc length differential is given by $ds = |r'(t)||dt|$, thus

$$\int_C f(x, y)\, ds = \int_a^b f(r)\, |r'(t)| dt = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

We note that since $a \le t \le b$, then we have $|dt| = dt$.

Definition 6.1.2 Given a smooth curve $C: r(t) = x(t)i + y(t)j, \quad a \le t \le b$.

$$\int_C f(x, y)\, dx = \int_a^b f\big(x(t), y(t)\big)\, x'(t) dt, \qquad \int_C f(x, y)\, dy = \int_a^b f\big(x(t), y(t)\big)\, y'(t) dt$$

are called the **line integrals** of f along C with respect to x and y.

Sometimes, we refer to the original line integral of f along C, namely

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt,$$

as the line integral of f along C with respect to arc length.

Definition 6.1.3 Let F be a continuous vector field defined on a domain containing a smooth curve C given by a vector function $r(t), t \in [a, b]$. The line integral of F along the curve C is $\int_C F \cdot dr = \int_a^b F\,(r(t)) \cdot r'(t) dt$.

Remark:

(1) The line integral along C, denoted by

$\int_C \ [P(x,y)dx + Q(x,y)dy]$ or $\int_{(a_1,b_1)}^{(a_2,b_2)} [Pdx + Qdy]$

(this could be evaluate by the definite integral $\int_{a_1}^{a_2} \left[P(x,f(x))dx + Q\big(x,f(x)\big)f'(x)dx\right]$)

(2) We make the following abbreviation:

$$\int\limits_C P(x,y)\,dx + Q(x,y)dy = \int\limits_C P(x,y)\,dx + \int\limits_C Q(x,y)\,dy$$

Examples

1. Evaluate $\int_C (2 + x^2 y)\,ds$, where C is the upper half of the unit circle traversed in the counter clockwise sense.

   Solution: We may parameterize C by $x = \cos t$, $y = \sin t$, $t \in [0,\pi]$. Thus

$$\begin{aligned}\int_C (2 + x^2 y)\,ds &= \int_0^{\pi}(2 + \cos^2 t \sin t)\sqrt{\sin^2 t + \cos^2 t}\,dt \\ &= \int_0^{\pi}(2 + \cos^2 t \sin t)\,dt \\ &= \left[2t - \tfrac{1}{3}\cos^3 t\right]_0^{\pi} \\ &= 2\pi + \tfrac{2}{3}.\end{aligned}$$

2. Evaluate $\int_{(0,1)}^{(1,2)}[(x^2 - y)dx + (y^2 + x)dy]$ along

   (a) straight line from $(0,1)$ to $(1,2)$

   (b) straight line from $(0,1)$ to $(1,1)$ and then from $(1,1)$ to $(1,2)$.

   Solution:

   (a) The equation of the straight line given, $(0,1)$ to $(1,2)$ in xy-plane is $y = x + 1$, $dy = dx$. So we have

$$\begin{aligned}\int_{(0,1)}^{(1,2)}[(x^2 - y)dx + (y^2 + x)dy] &= \int_0^1 [x^2 - (x+1)]dx + [(x+1)^2 + x]dx \\ &= \int_0^1 (2x^2 + 2x)dx \\ &= \tfrac{5}{3}.\end{aligned}$$

   (b) Along the straight line $(0,1)$ to $(1,1)$, we have $y = 1$, $dy = 0$.

$$\int_0^1 [x^2 - 1]dx + [1 + x](0) = \int_0^1 (x^2 - 1)\,dx = -\tfrac{2}{3}.$$

   Along the straight line from $(1,1)$ to $(1,2)$, $x = 1$, $dx = 0$

$$\int_1^2 [1-y](0) + [y^2+1]dy = \int_1^2 (y^2+1)\,dy = \frac{10}{3}.$$

Then the required value $= -\frac{2}{3} + \frac{10}{3} = \frac{8}{3}$.

3. Evaluate $\int_C\ y^2dx + xdy$, where
   (a) $C = C_1$ is the line segment from $(-5,-3)$ to $(0,2)$,
   (b) $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$.

   Solution:

   (a) $C_1: x = 5t-5,\ y = 5t-3,\ 0 \le t \le 1$. Using

   $\int_C\ f(x,y,z)\,dz = \int_a^b f(x(t),y(t),z(t))\,z'(t)dt$, and remark number 2, we have

$$\int_{C_1} y^2dx + xdy = \int_0^1 (5t-3)^2\,5dt + \int_0^1 (5t-5)\,5dt = -\frac{5}{6}.$$

   (b) $C_2: x = 4-t^2,\ y = t, -3 \le t \le 2$. Thus

$$\int_{C_2} y^2dx + xdy = \int_{-3}^2 t^2(-2t)^2\,dt + \int_{-3}^2 (4-t^2)\,5dt = \frac{245}{6}\ .$$

4. Evaluate $\int_C\ F\cdot dr$, where $F(x,y,z) = \langle xy, yz, zx\rangle$, and C is the curve $r(t) = \langle t, t^2, t^3\rangle$, $t \in [0,1]$.

   Solution: First $r'(t) = \langle 1, 2t, 3t^2\rangle$. Thus

$$F\big(r(t)\big)\cdot r'(t) = \langle t\cdot t^2, t^2\cdot t^3, t^3\cdot t\rangle \cdot \langle 1,2t,3t^2\rangle = t^3 + 5t^6.$$

   Therefore,

$$\int_C\ F\cdot dr = \int_0^1 F(r(t)\cdot r'(t)dt = \int_0^1 (t^3+5t^6)dt = \frac{27}{28}.$$

Theorem 6.2 (Fundamental Theorem for Line Integrals)

Let C be a smooth curve given by $r(t), t \in [a,b]$. Let f be a function of two or three variables whose gradient ∇f is continuous. Then $\int_C\ F\cdot dr = f\big(r(b)\big) - f(r(a))$.

Proof

$$\int_C\ F\cdot dr = \int_C\ \nabla f(r(t))\cdot r'(t)dt$$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right) dt$$

$= \int_a^b \frac{d}{dt} f(r(t))\,dt$ by Chain rule

$= f\big(r(b)\big) - f(r(a))$ by fundamental Theorem of Calculus. ■

Example Consider the gravitational (force) field $F(r) = \frac{mMG}{|r|^3}$, where $r = \langle x, y, z \rangle$. Recall that $F = \nabla f$, where $f(x, y, z) = \frac{mMG}{\sqrt{x^2+y^2+z^2}}$. Find the work done by the gravitational field in moving a particle of mass m from the point $(3,4,12)$ to the point $(1,0,0)$ along a piecewise smooth curve C.

Solution: $Work\ done\ (W) = \int_C F \cdot dr = \int_C \nabla f \cdot dr = f(1,0,0) - f(3,4,12) = \frac{12mMG}{13}$.

Definition 6.3.1 A **simple** curve is a curve which does not intersect itself.

Definition 6.3.2 A subset D in $\mathbb{R}^n$ is said to be **connected** if any two points in D can be joined by a path that lies in D.

Theorem 6.3.3 (Green's Theorem)

Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If $P(x, y)$ and $Q(x, y)$ have continuous partial derivatives on an open simply connected region that contains D, then

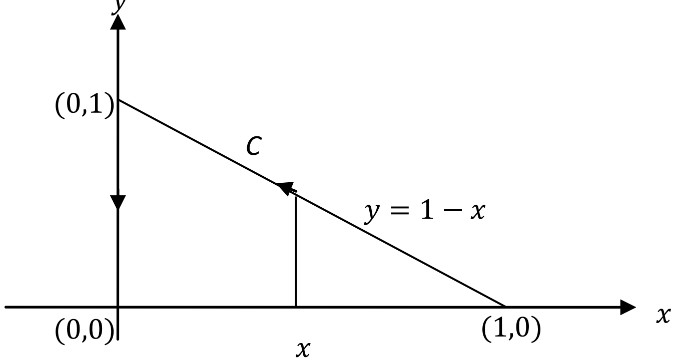
$$\int_C Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

The line integral $\int_C Pdx + Qdy$ has other notations as $\oint_C P\,dx + Qdy$, or $\oint_{\partial D} P\,dx + Qdy$. They all indicated the line integral is calculated using the positive orientation of C.

Examples

1. Evaluate $\int_C x^4\,dx + xydy$ where C is the triangular curve consisting of the line segment from $(0,0)$ to $(1,0)$, from $(1,0)$ to $(0,1)$.

   Solution: The function $P(x, y) = x^4$ and $Q(x, y) = xy$ have continuous partial derivatives on the whole of $\mathbb{R}^2$, which is open and simply connected.



By Green's Theorem,

$$\int_C x^4\,dx + xydy = \iint_D \left(\frac{\partial(xy)}{\partial x} - \frac{\partial(x^4)}{\partial y}\right) dA$$

$$= \iint_D y\,dydx$$

$$= \int_0^1 \int_0^{1-x} y\,dydx$$

$$= \frac{1}{6}.$$

2. Evaluate $\oint_C \left(3y - e^{\sin x}\right) dx + \left(7x + \sqrt{y^4+1}\right)dy$, where C is the circle $x^2+y^2=9$, oriented in the counter clockwise sense.

   Solution: C bounds the circular disk $D = \{(x,y)|x^2+y^2 \le 9\}$ and is given the positive orientation. By Green's Theorem,

$$\oint_C \left(3y - e^{\sin x}\right) dx + \left(7x + \sqrt{y^4+1}\right)dy = \iint_D \left[\frac{\partial(7x+\sqrt{y^4+1})}{\partial x} - \frac{\partial(3y-e^{\sin x})}{\partial y}\right] dA$$

$$= \iint_D 4\,dA$$

$$= 4(\pi 3^2)$$

$$= 36\pi.$$

[Green's theorem to find Area: Area of $D = \iint_D 1\,dA = \oint_{\partial D} P\,dx + Qdy$]

3. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

   Solution: Let the parametric equation for the ellipse be $x = a\cos t$, $y = b\sin t$ for $t \in [0,2\pi]$. Then

$$Area\ (A) = \frac{1}{2}\oint_{\partial D} x\,dy - ydx$$

$$= \frac{1}{2}\int_0^{2\pi} (a\cos t)\,(b\cos t) - (b\sin t)(-a\sin t)dt$$

$$= \frac{1}{2}\int_0^{2\pi} ab\,dt$$

$$= \pi ab.$$

4. Let $F(x,y) = \frac{-y}{x^2+y^2}i + \frac{x}{x^2+y^2}j$. Show that $\int_C F\cdot dr = 2\pi$ for every simple closed curve that encloses the origin.

Solution: We note that the vector field F is defined on $\mathbb{R}^2\backslash\{0,0\}$. Let C be any closed curve that enclose the origin. Choose a circle C^1 centred at the origin with a small radius a such that C^1 lies inside$C$. We can parameterize C^1, $x = a\cos t,\ y = a\sin t$, $t \in [0,2\pi]$. Let D be the region bound between C and C^1. We give both C and C^1 the counter clockwise orientation. Thus, $\partial D = C - C^1$ is given the positive orientation with respect to the region D. By Green's Theorem, we have

$$\int_{\partial D} F \cdot dr = \iint_D \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) dA$$

$$= \iint_D \frac{y^2-x^2}{(x^2+y^2)^2} - \frac{y^2-x^2}{(x^2+y^2)^2} dA = 0.$$

Thus,

$$\int_{\partial D} F \cdot dr = \int_{C^1} F \cdot dr$$

$$= \int_0^{2\pi} F\,(r(t)) \cdot r'(t) dt$$

$$= \int_0^{2\pi} \frac{(-a\sin t)(-a\sin t)+(a\cos t)(a\cos t)}{(a^2\cos^2 t + a^2\sin^2 t)} dt$$

$$= 2\pi. \qquad \blacksquare$$

Exercise 5

1. Evaluate $\int_C\ 2x\, ds$, where C consists of the arc C_1 of parabola $y = x^2$ from $(0,0)$ to $(1,1)$ followed by the vertical line segment C_2 from $(1,1)$ to $(1,2)$. *Ans.:* $\frac{1}{6}\left(5\sqrt{5}+11\right)$
2. Evaluate by Green's Theorem, $\oint_C\ e^{-x}\sin y\, dx + e^{-x}\cos y\, dy$, where C is the rectangle with vertices $(0,0), (\pi, 0), \left(\pi, \frac{\pi}{2}\right), (0, \frac{\pi}{2})$, oriented in the counter clockwise sense. *Ans.:* $2(e^{-\pi} - 1)$