Vector Spaces

One of the fundamental concepts of linear algebra is the concept of vector space. At the same time it is one of the more often used concepts algebraic structure in modern mathematics. For example, many function sets studied in mathematical analysis are with respect to their algebraic properties vector spaces. In analysis the notion ``linear space'' is used unstead of the notion ``vector space''.

Definition 1.1.1 A set $\mathbf{X}$ is called a *vector space over the number field* $\mathbf{K}$, if to every pair $(\mathbf{x}, \mathbf{y})$ of elements of $\mathbf{X}$ there corresponds a sum $\mathbf{x} + \mathbf{y} \in \mathbf{X}$,, and to every pair $(\alpha, \mathbf{x})$, where $\alpha \in \mathbf{K}$ and $\mathbf{x} \in \mathbf{X}$,, there corresponds an element $\alpha\mathbf{x} \in \mathbf{X}$, with the properties 1-8:

1. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ (commutability of addition);

2. $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$ (associativity of addition);

3. $\exists \mathbf{0} \in \mathbf{X} : \mathbf{0} + \mathbf{x} = \mathbf{x}$ (existence of null element);

4. $\forall \mathbf{x} \in \mathbf{X} \Rightarrow \exists -\mathbf{x} \in \mathbf{X} : \mathbf{x} + (-\mathbf{x}) = \mathbf{0}$ (existence of the inverse element);

5. $1 \cdot \mathbf{x} = \mathbf{x}$ (unitarism);

6. $\alpha(\beta\mathbf{x}) = (\alpha\beta)\mathbf{x}$ (associativity with respect to number multiplication);

7. $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$ (distributivity with respect to vector additism);

8. $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$ (distributivity with respect to number additism).

The properties 1-8 are called the vector space axioms. Axioms 1-4 shows that $\mathbf{X}$ is a commutative group or an Abelian group with respect to vector addition. The second correspondence is called multiplication of the vector by a number, and it satisfies axioms 5-8. Elements of a vector space are called vectors. If $\mathbf{K} = \mathbf{R}$, then one speaks of a real vector space, and if $\mathbf{K} = \mathbf{C}$, then of a complex vector space. Instead of the notion

Vector Spaces

One of the fundamental concepts of linear algebra is the concept of vector space. At the same time it is one of the more often used concepts algebraic structure in modern mathematics. For example, many function sets studied in mathematical analysis are with respect to their algebraic properties vector spaces. In analysis the notion ``linear space'' is used unstead of the notion ``vector space''.

Definition 1.1.1 A set $\mathbf{X}$ is called a *vector space over the number field* $\mathbf{K}$, if to every pair $(\mathbf{x}, \mathbf{y})$ of elements of $\mathbf{X}$ there corresponds a sum $\mathbf{x} + \mathbf{y} \in \mathbf{X}$,, and to every pair $(\alpha, \mathbf{x})$, where $\alpha \in \mathbf{K}$ and $\mathbf{x} \in \mathbf{X}$,, there corresponds an element $\alpha\mathbf{x} \in \mathbf{X}$, with the properties 1-8:

1. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ (commutability of addition);

2. $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$ (associativity of addition);

3. $\exists \mathbf{0} \in \mathbf{X} : \mathbf{0} + \mathbf{x} = \mathbf{x}$ (existence of null element);

4. $\forall \mathbf{x} \in \mathbf{X} \Rightarrow \exists -\mathbf{x} \in \mathbf{X} : \mathbf{x} + (-\mathbf{x}) = \mathbf{0}$ (existence of the inverse element);

5. $1 \cdot \mathbf{x} = \mathbf{x}$ (unitarism);

6. $\alpha(\beta\mathbf{x}) = (\alpha\beta)\mathbf{x}$ (associativity with respect to number multiplication);

7. $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$ (distributivity with respect to vector additism);

8. $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$ (distributivity with respect to number additism).

The properties 1-8 are called the vector space axioms. Axioms 1-4 shows that $\mathbf{X}$ is a commutative group or an Abelian group with respect to vector addition. The second correspondence is called multiplication of the vector by a number, and it satisfies axioms 5-8. Elements of a vector space are called vectors. If $\mathbf{K} = \mathbf{R}$, then one speaks of a real vector space, and if $\mathbf{K} = \mathbf{C}$, then of a complex vector space. Instead of the notion ``vector space'' we shall use the abbreviative ``space''.

Example 1.1.1. Let us consider the set of all $n \times 1$–matrices with real elements:

$$\mathbf{X} = \{ \mathbf{x} : \mathbf{x} = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \wedge \xi_i \in \mathbf{R} \}.$$

The sum of two matrices we define in usual way by the addition of the corresponding elements. By multiplying the matrix by a real number λ we multiply all elements of the matrix by this number. The simple check will show that conditions 1-8 are satisfied. For example, let us check conditions 3 and 4. We construct

$$\mathbf{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \quad -\mathbf{x} = \begin{bmatrix} -\xi_1 \\ \vdots \\ -\xi_n \end{bmatrix}.$$

As

$$\mathbf{0} + \mathbf{x} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} = \begin{bmatrix} 0+\xi_1 \\ \vdots \\ 0+\xi_n \end{bmatrix} = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} = \mathbf{x},$$

the element $\mathbf{0}$ satisfied condition 3 for arbitrary $\mathbf{x} \in \mathbf{X}$, and thus it is the null element of the space $\mathbf{X}$. For the element $-\mathbf{x}$

$$\mathbf{x} + (-\mathbf{x}) = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} + \begin{bmatrix} -\xi_1 \\ \vdots \\ -\xi_n \end{bmatrix} = \begin{bmatrix} \xi_1 - \xi_1 \\ \vdots \\ \xi_n - \xi_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{0},$$

i.e., condition 4 is satisfied. Make sure of the valitidy of the remaining conditions 1-2 and 5-8.

The vector space in example 1.1.1 is called *n-dimensional real arithmetical space* or in short space $\mathbf{R}^n$. Declaring the vector $\mathbf{x}$ of the space $\mathbf{R}^n$ we often use the transposed matrix

$$\mathbf{x} = \begin{bmatrix} \xi_1 & \ldots & \xi_n \end{bmatrix}^T.$$

In this presentation we often use punctuation mark (comma, semicolon) to separate the components of the vector, for example

$$\mathbf{x} = \begin{bmatrix} \xi_1, & \ldots & ,\xi_n \end{bmatrix}^T.$$

Example 1.1.1.* Let U, be a set that consists of all pairs of real numbers $\mathbf{a} = (\alpha_1, \alpha_2)$, $\mathbf{b} = (\beta_1, \beta_2)$, ...We define addition and multiplication by scalar in U as follows:

$$\mathbf{a} + \mathbf{b} = ((\alpha_1^3 + \beta_1^3)^{1/3}, (\alpha_2^3 + \beta_2^3)^{1/3}),$$

$$\alpha \mathbf{a} = (\alpha\alpha_1, \alpha\alpha_2).$$

Is the set U a vector space?

Proposition 1.1.1. Let $\mathbf{X}$ be a vector space. For arbitrary vectors $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and number $\lambda \in \mathbf{K}$ the following assertions and equalities are valid:

- the null vector $\mathbf{0}$ of the vector space $\mathbf{X}$ is unique;
- the inverse vector $-\mathbf{x}$ of each $\mathbf{x} \in \mathbf{X}$ is unique;
- the uniqueness of the inverse vector allows to define the operation of subfraction by
$$\mathbf{x} - \mathbf{y} \stackrel{def}{=} \mathbf{x} + (-\mathbf{y});$$
- $\mathbf{x} = \mathbf{y} \Leftrightarrow \mathbf{x} - \mathbf{y} = \mathbf{0};$
- $0\mathbf{x} = \mathbf{0} \quad \forall \mathbf{x} \in \mathbf{X}$;
- $\lambda \mathbf{0} = \mathbf{0} \quad \forall \lambda \in K$;
- $(-1)\mathbf{x} = -\mathbf{x}$;
- $\lambda \mathbf{x} = \mathbf{0} \quad \Leftrightarrow \quad (\lambda = 0 \vee \mathbf{x} = \mathbf{0}).$

Become convinced of the trueness of these assertions! □

Example 1.1.2. Let us consider the set of all $(m \times n)$−matrices with complex elements. The sum of this matrices will be defined by the addition of the corresponding elements of the matrices. By multiplying the matrix by a complex number λ one will multiply by this number all the elements of the matrix. We leave the check that all conditions 1-8 are satisfied to the reader. This vector space over the complex number field $\mathbf{C}$ will be denoted $\mathbf{C}^{m \times n}$. If we confine ourselves to real matrices, then we shall get vector space $\mathbf{R}^{m \times n}$ over the number field $\mathbf{R}$. The space $\mathbf{C}^{m \times 1}$ will be identified with the space $\mathbf{C}^m$ and the space $\mathbf{R}^{m \times 1}$ with the space $\mathbf{R}^m$.

Example 1.1.3. The set $\mathbf{F}[\alpha, \beta]$ of all functions $\mathbf{x} : [\alpha, \beta] \rightarrow \mathbf{R}$ is a vector space (prove!) over the number field $\mathbf{R}$ if

$$(\mathbf{x} + \mathbf{y})(t) \stackrel{def}{=} \mathbf{x}(t) + \mathbf{y}(t) \quad \forall t \in [\alpha, \beta]$$

and

$$(\lambda \mathbf{x})(t) \stackrel{def}{=} \lambda \mathbf{x}(t) \quad \forall t \in [\alpha, \beta].$$

Subspaces of the Vector Space

Definition 1.2.1. The set $\mathbf{W}$ of vectors of the vector space $\mathbf{X}$ (over the field $\mathbf{K}$) that is a vector space will respect to vector addition and multiplication by a number defined in the vector space $\mathbf{X}$, is called a *subspace* of the vector space $\mathbf{X}$ and denoted $\mathbf{W} \leq \mathbf{X}$.

Proposition 1.2.1. The set $\mathbf{W}$ of vectors of the vector space $\mathbf{X}$ is a subspace of the vector space $\mathbf{X}$ if for each two vectors $\mathbf{x}, \mathbf{y} \in \mathbf{W}$ and each number $\lambda \in \mathbf{K}$ vectors $\mathbf{x} + \mathbf{y}$ and $\lambda\mathbf{x}$ belong to the set $\mathbf{W}$.

Proof. Necessity is obvious. To prove sufficiency, we have to show that in our case conditions 1-8 of the vector spaces are satisfied. Let us check condition 1. Let $\mathbf{x}, \mathbf{y} \in \mathbf{W} \subset \mathbf{X}$. By assumption, $\mathbf{x} + \mathbf{y} \in \mathbf{W} \subset \mathbf{X}$. As $\mathbf{X}$ is a vector space, then for $\mathbf{X}$ axiom 1 is satisfied, and then $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$. Therefore, for $\mathbf{W}$ axiom 1 is satisfied, too. Let us found the validity of condition 4. Let $\mathbf{x} \in \mathbf{W} \subset \mathbf{X}$. By assumption, $(-1)\mathbf{x} \in \mathbf{W} \subset \mathbf{X}$. On the other hand, by preposition 1, in $\mathbf{X}$ the equality $(-1)\mathbf{x} = -\mathbf{x}$ holds. Hence the inverse vector $-\mathbf{x}$ belongs to set $\mathbf{W}$ with the vector $\mathbf{x}$, i.e., condition 4 is satisfied. Prove by yourselves the validity of condition 2, 3 and 5-8. □

Example 1.2.1. The vector space $\mathbf{C}[\alpha, \beta]$ over $\mathbf{R}$ of all functions continouos on $[\alpha, \beta]$ (example 1.1.3) is a subspace of vector space $\mathbf{F}\,[\alpha, \beta]$. As the sum of two functions continouos on the interval, and the product of such a function by a number are functions continouos on this interval, by proposition 1.2.1, $\mathbf{C}[\alpha, \beta]$ is a subspace of the vector space $\mathbf{F}\,[\alpha, \beta]$.

Example 1.2.2. Let $\mathbf{P}_n$ be the set of all polynomials $a_0t^k + a_1t^{k-1} + \ldots + a_{k-1}t + a_k = \mathbf{x} \quad (k \leq n)$ of at most degree n with real coefficients. We define addition of two polynomials and multiplication of a polynomial by a real number in usual way. As the result, we get the vector space $\mathbf{P}_n$ of polynomials of at most degree n. If we denote by $\mathbf{P}_n[\alpha, \beta]$ the vector space of polynomials of at most degree n defined on the interval $[\alpha, \beta]$, then $\mathbf{P}_n[\alpha, \beta]$ will be a subspace of vector space $\mathbf{C}[\alpha, \beta]$.

Example 1.2.3.* Let us show that the set $\mathbf{H} = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbf{R} \right\}$ is a subspace in the matrix vector space $\mathbf{R}^{2\times 2}$.

The set $\mathbf{H}$ is closed with respect to additism and multiplication by scalar since

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} + \begin{bmatrix} d & e \\ 0 & f \end{bmatrix} = \begin{bmatrix} a+d & b+e \\ 0 & c+f \end{bmatrix}$$

and

$$\alpha \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ 0 & \alpha c \end{bmatrix}.$$

Thus the set $\mathbf{H}$ is a subspace in the matrix vector space $\mathbf{R}^{2\times 2}$.

Problem 1.2.1.* Prove that the set of all symmetric matrices form a subspace in the vector space of all square matrices $\mathbf{R}^{n\times n}$.

Proposition 1.2.2. If $\mathbf{S}_1, \ldots, \mathbf{S}_k$ are subspaces of the vector space $\mathbf{X}$, then the intersection $\mathbf{S} = \mathbf{S}_1 \cap \mathbf{S}_2 \cap \ldots \cap \mathbf{S}_k$ of the subspaces is a subspace of the vector space $\mathbf{X}$.

Prove! □

Proposition 1.2.3. If $\mathbf{S}_1, \cdots, \mathbf{S}_k$ are subspaces of the space $\mathbf{X}$ and

$$\mathbf{S} = \{\mathbf{x}_1 + \mathbf{x}_2 + \ldots + \mathbf{x}_k : \mathbf{x}_i \in \mathbf{S}_i \quad (i = 1 : k)\}$$

is the sum of these subspaces, then $\mathbf{S}$ is a subspace in $\mathbf{X}$.

Definition 1.2.2. If each $\mathbf{x} \in \mathbf{S}$ can be expressed uniquely in the form

$\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2 + \ldots + \mathbf{x}_k \quad (\mathbf{x}_i \in \mathbf{S}_i)$, then we say that $\mathbf{S}$ is the *direct sum of subspaces* $\mathbf{S}_i$, and it denoted $\mathbf{S} = \mathbf{S}_1 \oplus \mathbf{S}_2 \oplus \cdots \oplus \mathbf{S}_k$.

Definition 1.2.3. Each element of the space $\mathbf{X}$ that can be expressed as $\alpha_1\mathbf{x}_1 + \ldots + \alpha_n\mathbf{x}_n$, where $\alpha_i \in \mathbf{K}$, is called a *linear combination* of the elements $\mathbf{x}_1, \ldots, \mathbf{x}_n$ of the vector space $\mathbf{X}$ (over the field $\mathbf{K}$).

Definition 1.2.4. The set of all possible linear combination of the set Z is called the *span of the set* $Z \subset \mathbf{X}$.

degree n defined on the interval $[\alpha, \beta]$, then $\mathbf{P}_n[\alpha, \beta]$ will be a subspace of vector space $\mathbf{C}[\alpha, \beta]$.

Example 1.2.3.* Let us show that the set $\mathbf{H} = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbf{R} \right\}$ is a subspace in the matrix vector space $\mathbf{R}^{2\times 2}$.

The set $\mathbf{H}$ is closed with respect to additism and multiplication by scalar since

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} + \begin{bmatrix} d & e \\ 0 & f \end{bmatrix} = \begin{bmatrix} a+d & b+e \\ 0 & c+f \end{bmatrix}$$

and

$$\alpha \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ 0 & \alpha c \end{bmatrix}.$$

Thus the set $\mathbf{H}$ is a subspace in the matrix vector space $\mathbf{R}^{2\times 2}$.

Problem 1.2.1.* Prove that the set of all symmetric matrices form a subspace in the vector space of all square matrices $\mathbf{R}^{n\times n}$.

Proposition 1.2.2. If $\mathbf{S}_1, \ldots, \mathbf{S}_k$ are subspaces of the vector space $\mathbf{X}$, then the intersection $\mathbf{S} = \mathbf{S}_1 \cap \mathbf{S}_2 \cap \ldots \cap \mathbf{S}_k$ of the subspaces is a subspace of the vector space $\mathbf{X}$.

Prove! □

Proposition 1.2.3. If $\mathbf{S}_1, \cdots, \mathbf{S}_k$ are subspaces of the space $\mathbf{X}$ and

$$\mathbf{S} = \{\mathbf{x}_1 + \mathbf{x}_2 + \ldots + \mathbf{x}_k : \mathbf{x}_i \in \mathbf{S}_i \quad (i = 1 : k)\}$$

is the sum of these subspaces, then $\mathbf{S}$ is a subspace in $\mathbf{X}$.

Definition 1.2.2. If each $\mathbf{x} \in \mathbf{S}$ can be expressed uniquely in the form $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2 + \ldots + \mathbf{x}_k \quad (\mathbf{x}_i \in \mathbf{S}_i)$, then we say that $\mathbf{S}$ is the *direct sum of subspaces* $\mathbf{S}_i$, and it denoted $\mathbf{S} = \mathbf{S}_1 \oplus \mathbf{S}_2 \oplus \cdots \oplus \mathbf{S}_k$.

Definition 1.2.3. Each element of the space $\mathbf{X}$ that can be expressed as $\alpha_1\mathbf{x}_1 + \ldots + \alpha_n\mathbf{x}_n$, where $\alpha_i \in \mathbf{K}$, is called a *linear combination* of the elements $\mathbf{x}_1, \ldots, \mathbf{x}_n$ of the vector space $\mathbf{X}$ (over the field $\mathbf{K}$).

Definition 1.2.4. The set of all possible linear combination of the set Z is called the *span of the set* $Z \subset \mathbf{X}$.

Example 1.2.4. Let $\mathbf{X} = \mathbf{R}^3$ and $Z = \{ [1; 1; 0]^T, [1; -1; 0]^T \}$. Then $span\, Z = \{ [\alpha; \beta; 0]^T : \alpha, \beta \in \mathbf{R} \}$. Prove!

Proposition 1.2.4. The set $span\, Z$ of the set $Z \subset \mathbf{X}$ is the least subspace that contain the set Z.

Proof. First, let us prove that $span\, Z$ is a subspace of the space $\mathbf{X}$. It is sufficient, by proposition 1.2.1, to show that $span\, Z$ is closed with respect to vector addition and multiplication of the vector by a number:

$$\mathbf{x}, \mathbf{y} \in span\, Z \Leftrightarrow \mathbf{x} = \sum_{i=1}^{n} \alpha_i \mathbf{u}_i \wedge \mathbf{y} = \sum_{j=1}^{m} \beta_j \mathbf{v}_j \wedge \alpha_i, \beta_j \in \mathbf{K} \wedge \mathbf{u}_i, \mathbf{v}_j \in Z \Rightarrow$$

$$\mathbf{x} + \mathbf{y} = \sum_{i=1}^{n} \alpha_i \mathbf{u}_i + \sum_{j=1}^{m} \beta_j \mathbf{v}_j \wedge \alpha_i, \beta_j \in \mathbf{K} \wedge \mathbf{u}_i, \mathbf{v}_j \in Z \Leftrightarrow \mathbf{x} + \mathbf{y} \in span\, Z ;$$

$$\lambda \in \mathbf{K} \wedge \mathbf{x} \in span\, Z \Leftrightarrow \lambda \in \mathbf{K} \wedge \mathbf{x} = \sum_{i=1}^{n} \alpha_i \mathbf{u}_i \wedge \mathbf{u}_i \in Z \wedge \alpha_i, \lambda \in \mathbf{K} \Rightarrow$$

$$\lambda \mathbf{x} = \lambda \sum_{i=1}^{n} \alpha_i \mathbf{u}_i = \sum_{i=1}^{n} (\lambda_i \alpha_i) \mathbf{u}_i = \sum_{i=1}^{n} \beta_i \mathbf{u}_i \wedge \beta_i \in \mathbf{K} \wedge \mathbf{u}_i \in Z \Leftrightarrow \lambda \mathbf{x} \in span\, Z .$$

Thus, $span\, Z$ is a subspace of the space $\mathbf{X}$. Let us show that $span\, Z$ is the least subspace of the space $\mathbf{X}$ that contains the set Z. Let $\mathbf{Y}$ be some subspace of the space $\mathbf{X}$ for which $\mathbf{Z} \subset \mathbf{Y}$. As $Z \subset \mathbf{Y}$ and $\mathbf{Y}$ is a subspace, the arbitrary linear combination of the elements os the set Z belongs to the subspace $\mathbf{Y}$. Therefore, $span\, Z$ as the set of all such linear combinations belongs to the space $\mathbf{Y}$. □

Corollary 1.2.1. A subset $\mathbf{W}$ of the vector space $\mathbf{X}$ is a subspace if it coincides with its span, i.e., $\mathbf{W} \leq \mathbf{X} \Leftrightarrow \mathbf{W} = span \mathbf{W}$.

Problem 1.2.2.* Does the vector $\mathbf{d} = \begin{bmatrix} 8 & 7 & 4 \end{bmatrix}^T$ belong to the subspace $span\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$,, when

$$\mathbf{a} = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}^{\mathbf{T}}, \mathbf{b} = \begin{bmatrix} 2 & 3 & 1 \end{bmatrix}^{\mathbf{T}}, \mathbf{c} = \begin{bmatrix} 6 & 9 & 3 \end{bmatrix}^{\mathbf{T}}?$$

Linear Dependence of Vectors. Basis of the Vector Space.

Definition 1.3.1. A set of vectors

$$\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$$

of the vector space $\mathbf{X}$ (over the field $\mathbf{K}$) is said to be *linearly dependent* if $\exists\, \alpha_1, \ldots, \alpha_k \in \mathbf{K} \;\; : |\alpha_1| + \ldots + |\alpha_k| \neq 0 \wedge \alpha_1 \mathbf{x}_1 + \ldots + \alpha_k \mathbf{x}_k = 0.$

Definition 1.3.2. A set of vectors of the space $\mathbf{X}$ (over the field $\mathbf{K}$) is said to be *linearly independent* if it is not linearly dependent.

Example 1.3.1.* Let us check if the set $U = \{1 + x,\ x + x^2,\ 1 + x^2\}$ is linearly independent in the vector space $\mathbf{P}_n$ $(n \geq 2)$ of all polynomials of at most degree n with real coefficients.
Let us consider the equality

$$\alpha(1 + x) + \beta(x + x^2) + \gamma(1 + x^2) = 0.$$

It is well-known in algebra that a polynomial is identically null if all its coefficients are zeros. Thus we get the system

$$\begin{cases} \alpha + \gamma = 0 \\ \alpha + \beta = 0 \\ \beta + \gamma = 0 \end{cases}.$$

This system has only a trivial solution. The set U is linearly independent.

Problem 1.3.1.* Prove that each set of vectors that contains the null vector is linearly dependent.

Problem 1.3.2.* Prove that if the column-vectors of determinant are linearly dependent, then the determinant equals 0.

Definition 1.3.3. A subset $V = \{\mathbf{x}_{i_1}, \ldots, \mathbf{x}_{i_k}\}$ of the set $U = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ of vectors of the vector space $\mathbf{X}$ is called a *maximal linearly independent subset* if V is linearly independent and it is not a proper subset of any linearly independent subset of the set U.

Proposition 1.3.1. If V is a maximal linearly independent subset of the set U, then $span\, U = span\, V$.

Proof. As $V \subset U$, $span\ V \subset span\ U$, by the definition of the span. To prove our assertion, we have to show that $span\ V \supset span\ U$. Let, by antithesis, exist a vector $\mathbf{x}$ of the subspace $span\ U$ that does not belong to the subspace $span\ V$. Thus, the vector $\mathbf{x}$ cannot be expressed as a linear combination of vectors of V but can be expressed as a linear combination of vectors of U, when at least one vector $\mathbf{x}_j \in U$ is used, at which $\mathbf{x}_j \notin V$ and $\mathbf{x}_j$ is not expressable as a linear combination of vectors of V. Set $V \cup \{\mathbf{x}_j\} \subset U$ is linearly independent and contains the set V as a proper subset. Hence V is not the maximal linearly independent subset. We have got a contradiction to tha assumption. Thus $span\ V \supset span\ U$, Q.E.D. □

Definition 1.3.4. A set $B = \{\mathbf{x}_i\}_{i \in I}$ of vectors of the vector space $\mathbf{X}$ is called a *basis* of the vector space $\mathbf{X}$ if B is linearly independent and each vector $\mathbf{x}$ of the space $\mathbf{X}$ can be expressed as a linear combination of vectors of the set B, $\mathbf{x} = \sum_{i \in I} \alpha_i \mathbf{x}_i$, where coeffitients α_i $(i=1: n)$ are called *coordinates* of the vector $\mathbf{x}$ relative to the basis B.

Definition 1.3.5. If the number of vectors in the basis B of the vector space $\mathbf{X}$, i.e., the number of elements of the set I, is finite, then this number is called the *dimension of the vector space* $\mathbf{X}$ and denoted $\dim \mathbf{X}$, and the space $\mathbf{X}$ is called *finite-dimensional* or a *finite-dimensional vector space*. If the number of vectors in the basis B of the vector space $\mathbf{X}$ is infinite, then the vector space $\mathbf{X}$ is called *infinite-dimensional* or an *infinite-dimensional vector space*.

Proposition 1.3.2. A subset B of the vectors of the vector space $\mathbf{X}$ is a basis of the space iff it is the maximal linearly independent subset.

Example 1.3.2. Vectors

$$\mathbf{e}_k = [\underbrace{0; 0; \ldots; 0}_{k-1\ zeros}; 1; \underbrace{0; \ldots; 0}_{n-k\ zeros}]^T \quad (k = 1 : n)$$

form a basis in space $\mathbf{R}^n$. Let us check the validity of the conditions in definition 1.3.4. As

$$\sum_{k=1}^{n} \alpha_k \mathbf{e}_k = \mathbf{0} \Leftrightarrow [\alpha_1, \ldots, \alpha_n]^T = [0; \ldots; 0]^T \Leftrightarrow \sum_{k=1}^{n} |\alpha_k| = 0,$$

the vector system $\{\mathbf{e}_k\}_{k=1:n}$ is linearly independent, and, due to

$$[\alpha_1, \ldots, \alpha_n]^T = \sum_{k=1}^{n} \alpha_k \mathbf{e}_k,$$

arbitrary vector of the space $\mathbf{R}^n$ can be expressed as a linear combination of vectors $\mathbf{e}_k$.

Problem 1.3.3. Vector system

$$\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$$

forms a basis in space $\mathbf{R}^{2\times2}$.

Example 1.3.3. Vector system $\{1; t; t^2; \ldots; t^n\}$ forms a basis in vector space $\mathbf{P}_n$ of polynomials of at most degree n. Truely, the set $\{1; t; t^2; \ldots; t^n\}$ is linearly independent since

$\mathbf{x} = a_0 t^n + a_1 t^{n-1} + \ldots + a_{n-1} t + a_n = \mathbf{0} \Rightarrow a_k = 0 \ (k = 1 : n)$

and each vector of space $\mathbf{P}_n$ (i.e., arbitrary polynomial of at most degree n) can be expressed in the form

$$\mathbf{x} = a_0 t^n + a_1 t^{n-1} + \ldots + a_{n-1} t + a_n.$$

Definition 1.3.6. Two vector spaces $\mathbf{X}$ and $\mathbf{X}'$ are called *isomorphic*, if there exist a one-to-one correspondence between the spaces $\varphi : \mathbf{X} \to \mathbf{X}'$, such that

1) $\forall \mathbf{x}, \mathbf{y} \in \mathbf{X}$ $\quad \varphi(\mathbf{x} + \mathbf{y}) = \varphi(\mathbf{x}) + \varphi(\mathbf{y})$;
2) $\forall \mathbf{x} \in \mathbf{X}, \forall \alpha \in \mathbf{K}$ $\quad \varphi(\alpha \mathbf{x}) = \alpha \varphi(\mathbf{x})$.

Proposition 1.3.3. All vector spaces (over the same number field $\mathbf{K}$) of the same dimension are isomorphic.

EXAMPLE 4.4 Consider the vector space $V = \mathbf{P}_n(t)$ consisting of all polynomials of degree $\leq n$.

(a) Clearly every polynomial in $\mathbf{P}_n(t)$ can be expressed as a linear combination of the $n+1$ polynomials

$$1, \quad t, \quad t^2, \quad t^3, \quad \ldots, \quad t^n$$

Thus, these powers of t (where $1 = t^0$) form a spanning set for $\mathbf{P}_n(t)$.

(b) One can also show that, for any scalar c, the following $n+1$ powers of $t - c$,

$$1, \quad t-c, \quad (t-c)^2, \quad (t-c)^3, \quad \ldots, \quad (t-c)^n$$

(where $(t-c)^0 = 1$), also form a spanning set for $\mathbf{P}_n(t)$.

EXAMPLE 4.5 Consider the vector space $\mathbf{M} = \mathbf{M}_{2,2}$ consisting of all 2×2 matrices, and consider the following four matrices in $\mathbf{M}$:

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Then clearly any matrix A in $\mathbf{M}$ can be written as a linear combination of the four matrices. For example,

$$A = \begin{bmatrix} 5 & -6 \\ 7 & 8 \end{bmatrix} = 5E_{11} - 6E_{12} + 7E_{21} + 8E_{22}$$

Accordingly, the four matrices $E_{11}, E_{12}, E_{21}, E_{22}$ span $\mathbf{M}$.

4.5 Subspaces

This section introduces the important notion of a subspace.

DEFINITION: Let V be a vector space over a field K and let W be a subset of V. Then W is a *subspace* of V if W is itself a vector space over K with respect to the operations of vector addition and scalar multiplication on V.

The way in which one shows that any set W is a vector space is to show that W satisfies the eight axioms of a vector space. However, if W is a subset of a vector space V, then some of the axioms automatically hold in W, because they already hold in V. Simple criteria for identifying subspaces follow.

THEOREM 4.2: Suppose W is a subset of a vector space V. Then W is a subspace of V if the following two conditions hold:

(a) The zero vector 0 belongs to W.

(b) For every $u, v \in W, k \in K$: (i) The sum $u + v \in W$. (ii) The multiple $ku \in W$.

Property (i) in (b) states that W is *closed under vector addition*, and property (ii) in (b) states that W is *closed under scalar multiplication*. Both properties may be combined into the following equivalent single statement:

(b') For every $u, v \in W, a, b \in K$, the linear combination $au + bv \in W$.

Now let V be any vector space. Then V automatically contains two subspaces: the set $\{0\}$ consisting of the zero vector alone and the whole space V itself. These are sometimes called the *trivial* subspaces of V. Examples of nontrivial subspaces follow.

EXAMPLE 4.6 Consider the vector space $V = \mathbf{R}^3$.

(a) Let U consist of all vectors in $\mathbf{R}^3$ whose entries are equal; that is,

$$U = \{(a, b, c) : a = b = c\}$$

For example, $(1, 1, 1)$, $(-3, -3, -3)$, $(7, 7, 7)$, $(-2, -2, -2)$ are vectors in U. Geometrically, U is the line through the origin O and the point $(1, 1, 1)$ as shown in Fig. 4-1(a). Clearly $0 = (0, 0, 0)$ belongs to U, because

EXAMPLE 4.4 Consider the vector space $V = \mathbf{P}_n(t)$ consisting of all polynomials of degree $\leq n$.

(a) Clearly every polynomial in $\mathbf{P}_n(t)$ can be expressed as a linear combination of the $n+1$ polynomials

$$1, \quad t, \quad t^2, \quad t^3, \quad \ldots, \quad t^n$$

Thus, these powers of t (where $1 = t^0$) form a spanning set for $\mathbf{P}_n(t)$.

(b) One can also show that, for any scalar c, the following $n+1$ powers of $t-c$,

$$1, \quad t-c, \quad (t-c)^2, \quad (t-c)^3, \quad \ldots, \quad (t-c)^n$$

(where $(t-c)^0 = 1$), also form a spanning set for $\mathbf{P}_n(t)$.

EXAMPLE 4.5 Consider the vector space $\mathbf{M} = \mathbf{M}_{2,2}$ consisting of all 2×2 matrices, and consider the following four matrices in $\mathbf{M}$:

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Then clearly any matrix A in $\mathbf{M}$ can be written as a linear combination of the four matrices. For example,

$$A = \begin{bmatrix} 5 & -6 \\ 7 & 8 \end{bmatrix} = 5E_{11} - 6E_{12} + 7E_{21} + 8E_{22}$$

Accordingly, the four matrices E_{11}, E_{12}, E_{21}, E_{22} span $\mathbf{M}$.

4.5 Subspaces

This section introduces the important notion of a subspace.

DEFINITION: Let V be a vector space over a field K and let W be a subset of V. Then W is a *subspace* of V if W is itself a vector space over K with respect to the operations of vector addition and scalar multiplication on V.

The way in which one shows that any set W is a vector space is to show that W satisfies the eight axioms of a vector space. However, if W is a subset of a vector space V, then some of the axioms automatically hold in W, because they already hold in V. Simple criteria for identifying subspaces follow.

THEOREM 4.2: Suppose W is a subset of a vector space V. Then W is a subspace of V if the following two conditions hold:

(a) The zero vector 0 belongs to W.

(b) For every $u, v \in W, k \in K$: (i) The sum $u + v \in W$. (ii) The multiple $ku \in W$.

Property (i) in (b) states that W is *closed under vector addition*, and property (ii) in (b) states that W is *closed under scalar multiplication*. Both properties may be combined into the following equivalent single statement:

(b') For every $u, v \in W, a, b \in K$, the linear combination $au + bv \in W$.

Now let V be any vector space. Then V automatically contains two subspaces: the set $\{0\}$ consisting of the zero vector alone and the whole space V itself. These are sometimes called the *trivial* subspaces of V. Examples of nontrivial subspaces follow.

EXAMPLE 4.6 Consider the vector space $V = \mathbf{R}^3$.

(a) Let U consist of all vectors in $\mathbf{R}^3$ whose entries are equal; that is,

$$U = \{(a, b, c) : a = b = c\}$$

For example, $(1, 1, 1)$, $(-3, -3, -3)$, $(7, 7, 7)$, $(-2, -2, -2)$ are vectors in U. Geometrically, U is the line through the origin O and the point $(1, 1, 1)$ as shown in Fig. 4-1(a). Clearly $0 = (0, 0, 0)$ belongs to U, because all entries in 0 are equal. Further, suppose u and v are arbitrary vectors in U, say, $u = (a, a, a)$ and $v = (b, b, b)$. Then, for any scalar $k \in \mathbf{R}$, the following are also vectors in U:

$$u + v = (a+b,\ a+b,\ a+b) \qquad \text{and} \qquad ku = (ka,\ ka,\ ka)$$

Thus, U is a subspace of $\mathbf{R}^3$.

(b) Let W be any plane in $\mathbf{R}^3$ passing through the origin, as pictured in Fig. 4-1(b). Then $0 = (0, 0, 0)$ belongs to W, because we assumed W passes through, the origin O. Further, suppose u and v are vectors in W. Then u and v may be viewed as arrows in the plane W emanating from the origin O, as in Fig. 4-1(b). The sum $u + v$ and any multiple ku of u also lie in the plane W. Thus, W is a subspace of $\mathbf{R}^3$.

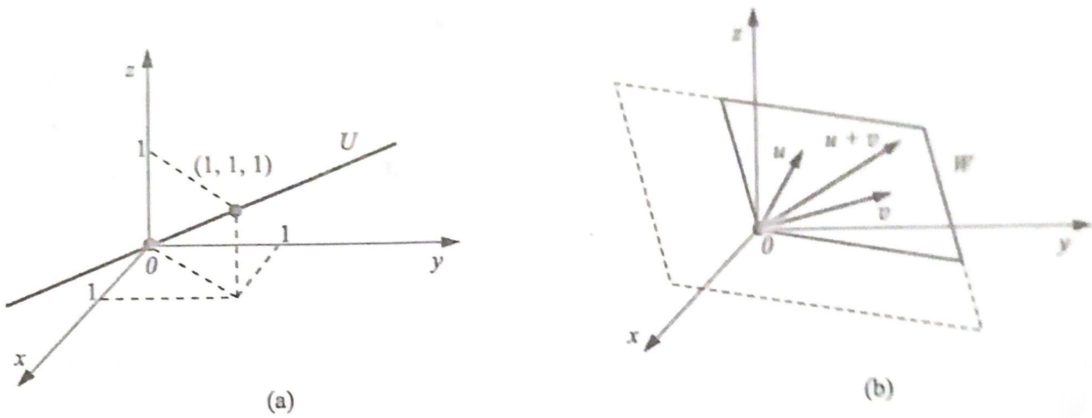

Figure 4-1

EXAMPLE 4.7

(a) Let $V = \mathbf{M}_{n,n}$, the vector space of $n \times n$ matrices. Let W_1 be the subset of all (upper) triangular matrices and let W_2 be the subset of all symmetric matrices. Then W_1 is a subspace of V, because W_1 contains the zero matrix 0 and W_1 is closed under matrix addition and scalar multiplication; that is, the sum and scalar multiple of such triangular matrices are also triangular. Similarly, W_2 is a subspace of V.

(b) Let $V = \mathbf{P}(t)$, the vector space $\mathbf{P}(t)$ of polynomials. Then the space $\mathbf{P}_n(t)$ of polynomials of degree at most n may be viewed as a subspace of $\mathbf{P}(t)$. Let $\mathbf{Q}(t)$ be the collection of polynomials with only even powers of t. For example, the following are polynomials in $\mathbf{Q}(t)$:

$$p_1 = 3 + 4t^2 - 5t^6 \qquad \text{and} \qquad p_2 = 6 - 7t^4 + 9t^6 + 3t^{12}$$

(We assume that any constant $k = kt^0$ is an even power of t.) Then $\mathbf{Q}(t)$ is a subspace of $\mathbf{P}(t)$.

(c) Let V be the vector space of real-valued functions. Then the collection W_1 of continuous functions and the collection W_2 of differentiable functions are subspaces of V.

Intersection of Subspaces

Let U and W be subspaces of a vector space V. We show that the intersection $U \cap W$ is also a subspace of V. Clearly, $0 \in U$ and $0 \in W$, because U and W are subspaces; whence $0 \in U \cap W$. Now suppose u and v belong to the intersection $U \cap W$. Then $u, v \in U$ and $u, v \in W$. Further, because U and W are subspaces, for any scalars $a, b \in K$,

$$au + bv \in U \qquad \text{and} \qquad au + bv \in W$$

Thus, $au + bv \in U \cap W$. Therefore, $U \cap W$ is a subspace of V.

The above result generalizes as follows.

THEOREM 4.3: The intersection of any number of subspaces of a vector space V is a subspace of V.