Outline

Elements of probability and probability distributions: Normal, Binomial, Poisson, Geometric, Hyper-geometric, Negative Binomial distributions.

Probability

Probability is the measure of the chance or likelihood that an event will occur. If (A) is an event within a sample space (S), then the probability of (A), denoted by (P(A)), is defined as the ratio of the number of outcomes favorable to (A) to the total number of possible outcomes in (S).

$$P(A) = \frac{n(A)}{n(S)}$$

Experiment: Is any operation which when performed generate a number of outcomes which cannot be predetermined. E.g Tossing two coins once.

Sample Space S : This is the totality of possible outcomes of a random experiment. For instance, in the experiment of tossing a coin, $S = (H, T)$

Event: This is a subset of a sample space. Suppose $S = (HH, HT, TH, TT)$, the event E of 2 heads occurring in the toss of a coin twice is $E = (HH)$

Null Event: An event that cannot occur is called a null event. For example, obtaining an orange fruit from a mango tree is a null event.

Sure Event: An event that is guaranteed to occur is called a sure event. For example, plucking an orange fruit from an orange tree is a sure event.

Mutually Exclusive Events: Events are said to be mutually exclusive, when they cannot happen together. In essence, the simultaneous occurrence of the events is doubtful. The event that a boy is playing football and table tennis at the same time is mutually exclusive.

Independent Events: Events are said to be independent if the occurrence or non-occurrence of one does not influence the occurrence or non-occurrence of the other. In other words, the events happen independently of each other.

Axiom of Probability

1. $P(A) \geq 0$
2. $P(S) = 1$
3. $0 \leq P(A) \leq 1$
4. $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i) \Rightarrow P(A_1 \cup A_2 \cup ...) = P(A_1) + P(A_2) + ...$

Fundamental Probability Rules

1. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
2. $P(A \cup B) = P(A) + P(B)$, if A and B are mutually exclusive
3. $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$
4. $P(A \cup B \cup C) = P(A) + P(B) + P(C)$, if A, B and C are mutually exclusive.
5. $P(A \cap B) = P(A) \times P(B)$, if A and B are independent
6. $P(A \cap B \cap C) = P(A) \times P(B) \times P(C)$, if A, B and C are independent
7. $P(A \cap B) = P(A / B)P(B)$

   $= P(B / A)P(A)$

8. $P(A^C) = 1 - P(A)$

Example 1

Find the probability of scoring a total of 7 points in a single toss of a pair of fair dice.

Solution

$$P(A) = \frac{\text{number of favourable outcomes}}{\text{Number of possible outcomes}}$$

For a pair of fair dice, possible outcomes = $6^2 = 36$

Total = 7 $\Rightarrow$ (1,6), (2,5) (3,4), (4,3), (5,2), (6,1).

$$P(\text{of 7 points}) = \frac{6}{36} = \frac{1}{6}$$

Example 2

One basket contains 4 oranges and 3 lemons; another basket contains 3 oranges and 2 lemons. If one fruit is drawn from each basket, find the probability that

a. both fruits are oranges.
b. both fruits are lemons.

Solution

a. P(both oranges) = $\frac{4}{7} \times \frac{3}{5} = \frac{12}{35}$ (both selections are independent).
b. P(both lemons) = $\frac{3}{7} \times \frac{2}{5} = \frac{6}{35}$ (both selections are independent).

Random Variables

A random variable is a function that assigns a real number to each outcome in the sample space, representing the possible results of an experiment.

Example 1

Let's consider the experiment of tossing 3 coins simultaneously.

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

Let X be the random variable representing the number of tails obtained when three coins are tossed simultaneously.

$x = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$

$= \{0, 1, 1, 1, 2, 2, 2, 3\}$

Let Y be the random variable representing the number of heads obtained when three coins are tossed simultaneously.

$y = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8\}$

$= \{3, 2, 2, 2, 1, 1, 1, 0\}$

Random variables can be classified into two types: discrete and continuous

Discrete Random Variables

A discrete random variable is one that assumes a finite or countably infinite set of distinct values. These values often whole numbers result from outcomes that can be individually counted, such as dice rolls, coin tosses, or card selections.

Continuous Random Variable

A continuous random variable can take any value within a specified interval, representing an uncountably infinite set of possible outcomes. Common examples include measurements such as height, weight, temperature, and time.

Expectation of a Random Variable

The mathematical expectation of a random variable denoted by E(X) is defined as

$$E(X) = \sum_{i=1}^{n} x_i p(x) \quad \text{for a discrete random variable}$$

$$= \int_{-\infty}^{\infty} x f(x) dx \quad \text{for a continuous random variable}$$

It should be noted that the mean of a random variable is the expectation of the random variable.

Variance of a random variable

The variance, V(X) of any random variable is defined as:

$V(X) = E(X^2) - [E(X)]^2$

Where $E(X^2) = \sum_{i=1}^{n} x_i^2 p(x)$ for a discrete random variable

$= \int_{-\infty}^{\infty} x^2 f(x)dx$ for a continuous random variable

Probability Distributions

Bernoulli Distribution

The Bernoulli distribution is a discrete probability distribution describing a random variable that takes the value 1 with probability (p) and 0 with probability (1 - p). It is used to model the outcome of a single trial with two possible results such as success or failure, yes or no, or heads or tails.

$$P(X=x) = \begin{cases} p^x(1-p)^{1-x}, x = 0,1 \\ 0, otherwise \end{cases}$$

Mean

$\mu = E(X)$

$= \sum_{x=0}^{1} xp(x)$

$= \sum_{x=0}^{1} x(p^x(1-p)^{1-x})$

$= 0 \times p^0 \times (1-p)^{1-0} + 1 \times p^1(1-p)^{1-1}$

$= 0 + p^1(1-p)^0$

$= 0 + p = p$

$\therefore \mu = p$

Variance

V(X) = E(X^2) -[E(X)]^2

$$E(X^2) = \sum_{x=0}^{1} x^2 p(x)$$

$$= \sum_{x=0}^{1} x^2 (p^x (1-p)^{1-x})$$

$$= 0^2 \times p^0 \times (1-p)^{1-0} + 1^2 \times p^1 \times (1-p)^{1-1}$$

$$= p^1 (1-p)^0 = p$$

$$V(X) = p - p^2$$

$$= p(1-p) = pq$$

Binomial Distribution

The binomial distribution is a discrete probability distribution characterized by two parameters: n (the number of independent trials) and p (the probability of success in each trial). A random variable X follows a binomial distribution if its probability density function is defined as

$$P(X=x) = \begin{cases} {}^nC_x \, p^x q^{n-x}, x = 0,1,2,...,n \\ 0, otherwise \end{cases}$$

Properties of Binomial Distribution

1. It has n independent trials.

2. It has constant probability of success P and probability of failure q =1-P

3. There is assigned probability to non-occurrence of events.

4. Each trial can result in one of only 2 possible outcomes called success or failure.

Mean and Variance of a Binomial Distribution

$$P(x) = {}^nc_x \, p^x q^{n-x}; \quad x = 0,1,2,.........., n$$

Mean:

$$E(X) = \sum_{x=0}^{n} x \; p(x)$$

$$= \sum_{x=0}^{n} x\, {}^{n}C_{x} p^{x} q^{n-x}$$

$$= \sum_{x=0}^{n} x \frac{n!}{(n-x)!x!} p^{x} q^{n-x}$$

$$= \sum_{x=0}^{n} x \frac{n(n-1)!}{(n-x)!x(x-1)!} p^{x} q^{n-x}$$

$$= n\sum_{x=1}^{n-1} \frac{(n-1)!}{(n-x)!(x-1)!} p^{1} p^{x-1} q^{n-x}$$

$$= np \sum_{x=1}^{n-1} \frac{(n-1)!}{(n-x)!(x-1)!} p^{x-1} q^{n-x}$$

Let s = x – 1, x = s + 1

$$= np \sum_{s=0}^{n-1} \frac{(n-1)!}{(n-s-1)!s!} p^{s} q^{n-s-1}$$

$$= np \sum_{s=0}^{n-1} \binom{n-1}{s} p^{s} q^{n-s-1}$$

$$= np\,(p+q)^{n-1} = np$$

Variance

$Var(X) = E(X^2) – [E(X)]^2$

$E[X^2] = E[X(X-1)] + E(X)$

$$E\ X(X-1) = \sum x(x-1)\,p(x)$$

$$= \sum_{x=0}^{n} x\,(x-1) \binom{n}{x} p^{x} q^{n-x}$$

$$= \sum x(x-1) \frac{n!}{(n-x)x!} p^{x} q^{n=x}$$

$$= \sum_{x=0}^{n} x(x-1) \frac{n(n-1)(n-2)!}{(n-x)!x(x-1)(x-2)!} p^{2} p^{x=2} q^{n-x}$$

$$= n(n-1)p^{2} \sum_{x=2}^{n-2} \frac{(n-2)!}{(n-x)!(x-2)!} p^{x=2} q^{n-x}$$

Let s = x-2 x = s + 2

$$= n(n-1)\,p^{2} \sum_{s=0}^{n-2} \frac{(n-2)!}{(n-s-2)!s!} p^{s} q^{n-s-2}$$

$$= n(n-1)\,p^{2} \sum_{s=0}^{n-2} \binom{n-2}{s} p^{s} q^{n-s-2}$$

$$E[X(X-1)] = n(n-1)p^{2}$$

$$\therefore\ E(X^{2}) = E[X(X-1)] + E(X)$$

$$= n(n-1)p^2 + np$$

$$\therefore\ V(X) = E(X^2) - [E(X)]^2$$
$$= n(n-1)\,p^2 + np - n^2p^2$$
$$= n^2p^2 - np^2 + np - n^2p^2$$
$$= np - np^2$$
$$= np\,(1-p)$$
$$= npq$$

Example

Past experience has shown that 7% of all luncheon vouchers are in error. If a random sample of 5 vouchers is selected, what is the probability that:

a) Exactly two will have error.

b) At least two will have error.

c) At most one will have error

Solution

n = 5, P = 7% = 0.07, q= 1- P = 1- 0.07 = 0.93

Let 'x' be the number of luncheon vouchers having error, then

$$P(X=x) = \binom{n}{x} p^x q^{n-x}; \quad x = 0,1,2,...,5$$

a) P(Exactly two will have error) = P(X=2)

$$\binom{5}{2}(0.07)^2(0.93)^{5-2} = 10(0.07)^2(0.93)^3 = 10\times 0.0049\times 0.804357 = 0.03941$$

b) P(At least two will have error) = $P(X\geq 2) = 1 - P(X\leq 1) = 1 - \ P(X=0) + P(X=1)$

$$= 1 - \left\{\binom{5}{0}(0.07)^0(0.93)^{5-0} + \binom{5}{1}(0.07)^1(0.93)^{5-1}\right\}$$

$$= 1 - \ 1(0.07)^0(0.93)^5 + 5(0.07)^1(0.93)^4$$

= 1 – {0.69569 + 0.26182}

= 1 – 0.95751 = 0.04249

c) P(At most one will have error) = $P(X\leq 1) = P(X=0) + P(X=1)$

$$= \binom{5}{0}(0.07)^0(0.93)^{5-0} + \binom{5}{1}(0.07)^1(0.93)^{5-1}$$

= 0.69569 + 0.26182
= 0.95751

Example 3

Suppose it is known from past experience that approximately 90% of human beings contracting HIV Virus die within 10 years of infection. Given that 5 people in a certain town contract the disease, use the binomial probability distribution to find the probability that exactly 4 of them will die within 10 years.

Solution

n=5, p = 90% = 0.9, p(4) = ?

$$P(4) = \binom{5}{4}(0.9)^4(1-0.9)^{5-4} = 0.32805 \sim 0.33$$

Poisson distribution

The Poisson distribution is a discrete probability distribution used to model the number of events occurring within a fixed interval of time or space. If X is a random variable following a Poisson distribution with an average rate λ, then the probability of observing exactly x events in that interval is given by:

$$p(x) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0,1,2,...$$

Properties of Poisson distribution

1. Poisson experiment results in outcomes that can be classified as success or failures.

2. The average numbers of success λ that occurs in a specified region is known.

3. The probability that a success will occur in an extremely small region is virtually zero.

The mean of the Poisson distribution

μ = E(X)

$$= \sum_{x=0} xp(x)$$

$$= \sum_{x=0}^{\infty} \frac{xe^{-\lambda}\lambda^x}{x!}, x = 0,1,2,...$$

$$= \sum_{x=1}^{\infty} \frac{xe^{-\lambda}\lambda^x}{x!}$$

$$= \lambda\sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!}$$

$$= \lambda e^{-\lambda} e^{\lambda} = \lambda$$

$$\therefore \mu = E(X) = \lambda$$

The Variance of the Poison Distribution

$V(X) = E(X^2) - [E(X)]^2$

$V(X) = E[X(X-1)] + E(X)$

$$E[X(X-1)] = \sum_{x=0}^{\infty} \frac{x(1-x)e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=2}^{\infty} \frac{x(1-x)e^{-\lambda}\lambda^x}{x!}$$

$$= \lambda^2 \sum_{x=2}^{\infty} \frac{e^{-\lambda}\lambda^{x-2}}{(x-2)!}$$

$$= \lambda^2 e^{-\lambda} e^{\lambda} = \lambda^2$$

Note that $\sum_{x=2}^{\infty} \frac{e^{-\lambda}\lambda^{x-2}}{(x-2)!} = 1$

$$\therefore V(X) = \lambda^2 + \lambda - \lambda^2 = \lambda$$

Example

Five percent of the inhabitants of a village who were attacked by cholera died. Find the probability that out of a sample of 60 cholera patients selected at random in the village, (a) exactly two (b) at least two will die.

Solution

P= 5% = 0.05, n= 60, λ = np= 60x0.05 = 3

(a) $p(2) = \frac{3^2 e^{-3}}{2!} = 0.2241 \cong 0.22$

(b) $p(x \geq 2) = 1 - p(x < 2) = 1 - [p(0) + p(1)] = 1 - [e^{-3} + 3e^{-3}]$

$= 1 - 0.1992 = 0.8008 \cong 0.8$

Hyper Geometric Distribution

A random variable X has a hyper geometric distribution, and it is referred to as a hyper geometric random variable, if and only if, its probability distribution is given by

$$p(X = x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}, x = 0, 1, 2, \ldots, n$$

The mean and variance of the hyper geometric distribution

The mean and variance shall be stated without proof. The mean of the hyper geometric distribution, μ = E(X) = $\frac{nk}{N}$ and the variance, V(X) = $\frac{nk(N-k)(N-n)}{N^2(N-1)}$

Example

Out of 10 applicants for a foreign scholarship, 5 are Nigerians. If 5 applicants are awarded the scholarships at random, find the probability that

a. all the 5 are Nigerians

b. only one is a Nigerian

Solution

a. N = 10, K = 5, n = 5, N-k = 5, x = 5

$$p(X = 5) = \frac{\binom{5}{5}\binom{5}{0}}{\binom{10}{5}} = 0.004$$

b. N= 10, K = 5, n = 5, N-k=5, x= 1

$$p(X = 1) = \frac{\binom{5}{1}\binom{5}{4}}{\binom{10}{5}} = 0.099$$

Negative Binomial Distribution

The Negative Binomial distribution is a discrete probability distribution that models the number of failures observed before achieving a specified number of successes in a sequence of independent Bernoulli trials, each with success probability (p). If a random variable (X) follows a Negative Binomial distribution with parameters (r) (number of successes) and (p) (success probability), then its probability mass function is:

$$P\ x\ = {}^{r+x-1}C_x p^r q^x ,\ x = 0,1,2,...$$

Mean

$$E(x) = \frac{rq}{p}$$

Variance

$$\text{Var(x)=}\frac{rq}{p^2}$$

Geometric distribution

The Geometric distribution is a discrete probability distribution that describes the number of failures observed before the first success in a sequence of independent Bernoulli trials, each with success probability (p). If a random variable (X) follows a Geometric distribution with parameter (p), its probability mass function is:

$$P(x) = (1-p)^x p; \qquad x = 0,1,2,\ldots$$

Mean

$$E(x) = \frac{1}{p}$$

Variance

$$\text{Var(x)} = \frac{q}{p^2}$$

Normal Distribution

Normal distribution also known as Gaussian distribution is a continuous probability distribution. A continuous random variable X is said to be normal distribution with parameter μ and σ^2 if the density function is given by

$$p(x) = \frac{e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}}{\sigma\sqrt{2\pi}}, -\infty < x < \infty$$

Properties of the Normal Distribution

1. The curve of a normal distribution is bell shaped
2. The mean, median and mode of the normal distribution coincide (i.e mean=median=mode) because the distribution is symmetrical and single peaked.
3. Areas under the curve can be approximated as follows:

   a 68.27% of this area is between $\mu \pm \sigma$ i.e.

   1 standard deviation from the mean.

   b. 95.45% of this area falls within $\mu \pm 2\sigma$ i.e

   2 standard deviations from the mean.

   c. 99.73% of this area falls within $\mu \pm 3\sigma$ i.e

   3 standard deviations from the mean.

Rather than using the raw value of the normal variable, X, it is more convenient to use its standardized form given as

$$z = \frac{x - \mu}{\sigma}$$

This means that X has mean μ and standard deviation σ but the variable Z has mean 0 and standard deviation 1. Z is the standard normal variate or score. The standard normal curve for different values of Z is tabulated for use and is called the table of standard normal distribution.

Mean

$E(x) = \mu$

Variance

$\text{Var(x)} = \sigma^2$

Example 2

The mean weight of cocoa produced by 20 farmers is 750 kg and the standard deviation is 65kg. If the weight is normally distributed, find how many farmers produce (a) less than 650kg (b) more than 780kg.

Solution

a. μ = 750, σ = 65, X = 650,

P (X< 650)

$$z = \frac{x-\mu}{\sigma} = \frac{650-750}{65} = -1.54$$

P (X< 650) = P (Z <-1.54) = P (Z >1.54)

= 0.0606 (check table)

b. P(X >780)

$$z = \frac{x-\mu}{\sigma} = \frac{780-750}{65} = 0.46$$

P(X >780) = P (Z > 0.46)

= 0.3102 (check table).