# BINOMIAL EXPANSION, THEOREM APPLICATION AND EXAMPLES

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**Definition:** A binomial is an expression of the form $(a+b)^n$ where $a, b \in \mathbb{R}$ $and$ $n \in \mathbb{Z}$.

- Question: How do we expand a binomial?

There are two ways to it;

# 1. BINOMIAL EXPANSION BY USING PASCAL TRIANGLE

<u>Preamble:</u>

Blaise Pascal ( 19/06/1623 – 19/08/1662), a French Mathematician and a philosopher observed that a table of coefficient (know as Pascal triangle) to determine the coefficient of the expansion

Suppose we consider the binomial $(a+b)^n$,

if $n = 0$ then $(a+b)^0 = 1$

n=1 implies $(a+b)^1 = a+b$

n=2 gives $(a+b)^2 = a^2 + 2ab + b^2$

n=3 $gives$ $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

n= $4$ $leads\ to$

$$a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

and so on.

Obviously , the coefficients of the expansions can be presented as a table in the next slide

# 2. BY USING BINOMIAL THEOREM

- For any real numbers $a, b$ and any natural number $n$, the following formula holds true.

$(a+b)^n$

$= {}^nC_0 a^n + {}^nC_1 a^{n-1} b + \cdots + {}^nC_r a^{n-r} b^k + {}^nC_n l$

$= \sum_k^n {}^nC_k a^{n-k} b^k$

$= \sum_k^n \frac{n!}{r!(n-r)!} a^{n-k} b^k$

# PROOF

- We prove by mathematical Induction

Let $n = 1$, then $(a+b)^1 = a^1 + b^1$ LHS =RHS

Hence, the formula is true .

Assume $n = k$ is true

That is $(a+b)^n = \sum_{r=0}^{k} k_{C_r} a^{k-r} b^r, k < n$

We verify that the formula is true for $k = n$

- $$(a+b)^n = (a+b)(a+b)^{n-1}$$

$$= (a+b) * \sum_{r=0}^{n-1} n-1_{C_r}\, a^{n-1-r} b^r$$

$$= \sum_{r=0}^{n-1} n-1_{C_r}\, a^{n-r} b^r + \sum_{r=1}^{n-1} n-1_{C_r}\, a^{n-r} b^r$$

$$= a^n + \sum_{r=1}^{n-1} \left(n-1_{C_r} + n-1_{C_{r-1}}\right) a^{n-r} b^r + b^n$$

$$= a^n + \sum_{r=1}^{n-1} n_{C_r}\, a^{n-r} b^r + b^n$$

$$= \sum_{r=1}^{n-1} n_{C_r}\, a^{n-r} b^r \ \left(since\, n_{C_0} = 1, n_{C_n} = 1\right).$$

Hence the formula holds true for $k = n$.

*Therefore, by induction the formula holds true for all positive integers n*

# Worked Examples

Example 1: State the binomial expansion of the theorem, when

$$a = 1 \; and \; b = x.$$

Solution :

Recall : Binomial theorem

$$(a+b)^n = \sum_{k=0}^{n} n_{C_k} \, a^{n-k} b^k$$

If $a = 1, b = x \; then$

$$(1+x)^n = \sum_{k=0}^{n} n_{C_k} \, 1^{n-k} x^k = \sum_{k=0}^{n} n_{C_k} \, x^k$$

$$= n_{C_0} \, x^0 + n_{C_1} \, x^1 + n_{C_2} \, x^2 + \ldots + n_{C_n} \, x^n$$

$$= 1 + n_{C_1} \, x^1 + n_{C_2} \, x^2 + \ldots + n_{C_n} \, x^n$$

# Example 2: Expand $(3-2x)^4$

- $(a+b)^n = n_{C_0}a^n + n_{C_1}a^{n-1}b^1 + n_{C_1}a^{n-2}b^2 + \ \ldots + n_{C_n}b^n$

$(3-2x)^4 = 3^4 + 4 * 3^3(-2x) + 6 * 3^2(-2x)^2 + 4 * 3^1(-2x)^3 + (-2x)^4$

$(3-2x)^4 = 81 - 216x + 216x - 96x^3 + 16x^4$

# Binomial Expansion for negative Integers

- Recall:

$$(a+b)^n = \sum_{r=0}^{n} n_{C_r} a^{n-r} b^r$$

Suppose $a = 1, b = x$ then we have

$$(1+x)^n$$

$$= 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots$$

$$+ \frac{n(n-1)\ldots(n-k+1)}{k!}.x^k + \cdots + x^n.$$

Example 3: Expand $\frac{}{(1+x^2)(1-2x)}$ to the ascending power of $x^2$.

# Evaluation with binomial expansion

Example 4: Evaluate $\left(\frac{1}{28}\right)^{\frac{1}{3}}$ to four significant figures.

Solution:

$$\left(\frac{1}{28}\right)^{\frac{1}{3}} = \frac{1}{(28)^{\frac{1}{3}}} = \frac{1}{(1+27)^{\frac{1}{3}}} = (1+27)^{\frac{-1}{3}} = (1+3^3)^{\frac{-1}{3}}$$

$$= \left(3^3\left(1+\frac{1}{3^3}\right)\right)^{\frac{-1}{3}} = \frac{1}{3}\left(1+\frac{1}{27}\right)^{\frac{-1}{3}}$$

By definition ,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots$$

$$\because \left(\frac{1}{28}\right)^{\frac{1}{3}} = \frac{1}{3}\left\{1 + \left(\frac{-1}{3}\right)\left(\frac{1}{27}\right) + \frac{\frac{-1}{3}\left(\frac{-1}{3}-1\right)}{2!}\left(\frac{1}{27}\right)^2 + \ \ldots\right\}$$

$$= \frac{1}{3}\left\{1 - \frac{1}{81} + \frac{2}{3^8} - \cdots\right\} = \frac{1}{3}\{0.9879\} \simeq 0.3293$$

# Largest Coefficient and Greatest Coefficient

- The general term $(rth\ term)$ of the binomial expansion of $(a+b)^n$ is given by:

$$T_r = n_{C_r a^r} b^{n-r} \ldots (1)$$

Suppose the binomial expression is of the form $(ax+by)^n$ then the $rth$ term becomes:

$$T_r = n_{C_r (ax)^r} (by)^{n-r} \ldots (2)$$

The Coefficient of $x^r$ in $(2)$ above is

$$K_r = n_{C_r a^r} (by)^{n-r} \ldots (3)$$

The $(r+1)th\ term\ gives$

$$K_{r+1} = n_{C_{r+1} a^{r+1}} (by)^{n-r-1} \ldots (4)$$

Algebraically: $k_{r+1} \geq k_r \ldots (5)$

and $k_r \leq k_{r+1} \ldots (6)$

Suppose the largest $r$ for which:

$$\left|\frac{k_r}{k_{r+1}}\right| \leq 1 \ \ or \left|\frac{k_{r+1}}{k_r}\right| \geq \ 1 \ \ \ldots \ \ (7)$$

If $r = m$, then the largest coefficient is $k_{m+1}$.

# Worked Examples

1) Expand $(2x + 1)^6$ and obtain the largest coefficient.
2) Obtain the largest equation of the expansion:
   a. $(3 + 4x)^5$
   b. $\left(7 + \frac{3}{2}x\right)^3$
3) Expand (up to $x^2$):
   a. $(3 + 4x)^{-2}(3 + 2x)$
   b. $\frac{3x-4}{(x+1)^2(x-2)}$

# Tutorial Questions

1. Show that $(1-2x^3)^7 = 1 - 14x^3 + 84x^6 - 280x^9 + 560x^{12} - 672x^{15} + 448x^{18} - 128x^{21}$.
2. Show that $(x - {}^1/_x)^5 = x^5 - 5x^3 + 10x - \frac{10}{x} + \frac{5}{x^3} - \frac{1}{x^5}$.
3. Obtain the coefficient of $x^4$ in the expansion of $(2-3x)^9$.
4. Evaluate $(1.03)^{10}$ to 3decimal places.
5. Expand $(1+y)^{10}$ in ascending powers of $y$ up to $y^4$.
6. Use the binomial expansion to evaluate $\frac{1}{\sqrt{17}}$ to four decimal places

## MATHEMATICAL INDUCTION

# Principle of **Mathematical Induction**

# Definition

Mathematical Induction (MI) is a technique or method of proving a statement/proposition that is asserted about every natural number.

MI is therefore, a tools/ technique for proving statement which in reality maybe too difficult to establish. It could be used to prove equations, theorems, formulas, inequalities and for solving problems in geometry.

For instance, let a proposition $T_n$ be

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

for $n = 1$; $1 = \frac{1(1+1)}{2} = \frac{1*2}{2} = 1$,

$$LHS = 1 = RHS$$

for $n = 2$; $1 + 2 = \frac{2(2+1)}{2} = \frac{2*3}{2} = 3$,

$$LHS = 3 = RHS$$

for $n = 3$; $1 + 2 + 3 = \frac{3(3+1)}{2} = \frac{3*4}{2} = 6$,

$$LHS = 6 = RHS$$

$$\vdots$$

for $n = 100$; $1 + 2 + 3 + \cdots + 100 = \frac{100(100+1)}{2} = \frac{100*101}{2}$,

$$LHS = RHS$$

In general,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

# Principle of Mathematical Induction

- To use MI to prove a proposition, four major steps are involved;

Let $T_n$ be the proposition, $n \in \mathbb{N}$

Step 1: We show that for $n = 1, T_1$ is either true or false.

Step 2: We assume true for $T_k$ if $T_1$ is true. Otherwise, we conclude $T_k$ is false. That is, for $\mathrm{n} = \mathrm{k}$, we assume that $T_k$ is true provided we already established that $T_1$ is true.

Step 3: We show that for $\mathrm{n} = \mathrm{k} + 1$, $T_{k+1}$ is either true or false.

Step 4: Conclusion

We conclude that since $T_1$ is either true or false, $T_k$ is assumed true, and $T_{k+1}$ is either true/false. Then the proposition $T_n$ is either true/ false for anv natural number $\mathrm{n} \in \mathbb{N}$.

# Worked Example

Prove that for any natural number $n \epsilon \mathbb{N}$

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Proof:

Let $T_n$ be the proposition that

$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

Step 1: We show that $T_1$ is either true or false $\text{LHS} = 1, RHS = \frac{1(1+1)}{2} = 1$

$LHS = RHS$, hence, $T_1$ is true.

Step 2: We assume that at $n = k$, $T_k$ is true.

That is,

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2} \quad \dots\dots\dots \quad (*)$$

Step 3: We show that $T_{k+1}$ is either true or false, that is, at $n = k + 1$

$$1 + 2 + 3 + \cdots + k + 1 = \frac{(k+1)(k+2)}{2} \quad \dots\dots\dots \quad (**)$$

Substituting $(*)$ into $(**)$, we obtain

$$\frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}$$

$$LHS = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}$$

$$= \frac{(k+1)(k+2)}{2} = RHS$$

Hence $LHS = RHS$ then the statement is true for $n = k + 1$ That is , $T_{k+1}$ is true.

Step 4: (Conclusion)

Since $T_1$ is true, $T_k$ is assumed true, and $T_{k+1}$ is true, then the proposition $T_n$ is true for all natural numbers

Hence,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \quad n \in \mathbb{N}$$

# Tutorial Questions

- Prove that:

1. $1 + 3 + 5 + \cdots + (2n + 1) = n^2, n \in \mathbb{N}$
2. $n! \le n^n$ for any integer $n \ge 1$
3. $(ab)^n = a^n b^n,\ n \in \mathbb{N}\quad a, b \in \mathbb{R}.$
4. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4},$ $n \epsilon \mathbb{N}$
5. $\frac{1}{1*3} + \frac{1}{3*5} + \frac{1}{5*7} + \ldots + \frac{1}{(2n-1)(2n+1)},$ $n \in \mathbb{N}$

5. $1^2 + 2^2 + 3^3 + \cdots + n^2 = \frac{n}{6}(n + 1)(2n + 1),\ n \epsilon \mathbb{N}$

Show that:

6. $2 + 4 + 6 + \cdots + 2n = n(n + 1),\ n\ \in \mathbb{N}$
7. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots + \frac{(-1)^{n-1}}{n}$ is always positive.
8. $a + (a + d) + (a + 2d) + (a + 3d) + \cdots + (a(n - 1)d) = \frac{n}{2}[2a + (n - 1)d]$.
9. $7^{2n} - 1$ is always a multiple of 8.